NOTES ON L. C. EVANS AND R. F. GARIEPY: *MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS*

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Notes on chapters 2, 3, and 5 of *Measure Theory and Fine Properties of Functions* by L. C. Evans and R. F. Gariepy. These notes cover topics from measure theory that are useful in PDE and the calculus of variations. We assume that the reader is already familiar with most topics offered in a standard first–semester graduate level course in measure theory and PDE, including differentiation, integration, and the theory of L^p spaces, as well as basic Sobolev space theory. All references are from eq:measure [1] unless indicated otherwise.

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1. GENERAL MEASURE THEORY

1.1. Weak Convergence and Compactness for Radon Measures.

t1.9-1 **Theorem 1.1.1.** *Let $\mu, \{\mu_k\}_{k=1}^{+\infty}$ be Radon measures on $\mathbb{R}^n$. The following three statements are equivalent:*

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(i) $\lim_{k\to+\infty}\int_{\mathbb{R}^n} f\,d\mu_k = \int_{\mathbb{R}^n} f\,d\mu$ *for all* $f \in \mathcal{C}_c(\mathbb{R}^n)$;

(ii) $\limsup_{k\to+\infty}\mu_k(K) \le \mu(K)$ *for each compact set* $K \subseteq \mathbb{R}^n$ *and* $\mu(U) \le \liminf_{k\to+\infty}\mu_k(U)$ *for each open set* $U \subseteq \mathbb{R}^n$;

(iii) $\lim_{k\to+\infty}\mu_k(B) = \mu(B)$ *for each bounded Borel set* $B \subseteq \mathbb{R}^n$ *with* $\mu(\partial B) = 0$.

Remark. *Recall that Radon measures on* $\mathbb{R}^n$ *are characterized by inner and outer regularity. Let* $B \subseteq \mathbb{R}^n$ *be a Borel set, and let* $K \subseteq B \subseteq U$ *with* K *compact and* U *open. If* $\{\mu_k\}_{k=1}^{+\infty}$ *is converging to* μ *in any sense, we should expect* $\mu_k(K) \le \mu(K)$ *for all* $k \in \mathbb{N}$ *and* $\mu_k(U) \ge \mu(U)$ *for all* $k \in \mathbb{N}$. *Conditions (ii) and (iii) tell us that this in fact holds up to a subsequence.*

Definition (Weak Convergence of Radon Measures)**.** *Let* $\mu, \{\mu_k\}_{k=1}^{+\infty}$ *be Radon measures on* $\mathbb{R}^n$. *We say that* $\{\mu_k\}_{k=1}^{+\infty}$ *converges weakly to* μ, *and write*

$$\mu_k \rightharpoonup \mu,$$

if

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} f\,d\mu_k = \int_{\mathbb{R}^n} f\,d\mu$$

for every $f \in \mathcal{C}_c(\mathbb{R}^n)$.

Proof. Assume first that (i) holds. Let $U \subseteq \mathbb{R}^n$ be open, and choose a compact set $K \subseteq U$. Next apply Urysohn's Lemma to choose a function $f \in \mathcal{C}_c(\mathbb{R}^n)$ such that

$$0 \le f \le 1, \quad \operatorname{supp}(f) \subseteq U, \quad \text{and} \quad f \equiv 1 \text{ on } K.$$

Then

$$\begin{aligned}\mu(K) &= \int_K d\mu = \int_K f\,d\mu \le \int_{\mathbb{R}^n} f\,d\mu = \lim_{k\to+\infty}\int_{\mathbb{R}^n} f\,d\mu_k \le \liminf_{k\to+\infty}\int_U d\mu_k \\ &= \liminf_{k\to\infty}\mu_k(U).\end{aligned}$$

Thus

$$\begin{aligned}\mu(U) &= \sup\{\mu(K) : K \text{ compact, } K \subseteq U\} \\ &\le \liminf_{k\to+\infty}\mu_k(U).\end{aligned}$$

This proves the second part of (ii). The first part is similar.

Next suppose that (ii) holds. Let $B \subseteq \mathbb{R}^n$ be a bounded Borel set, $\mu(\partial B) = 0$. Then by (ii),

$$\begin{aligned}\mu(B) = \mu(B^\circ) &\le \liminf_{k\to+\infty}\mu_k(B^\circ) \\ &\le \limsup_{k\to+\infty}\mu_k(\overline{B}) \\ &\le \mu(\overline{B}) \\ &= \mu(B).\end{aligned}$$

Since $\mu_k(B^\circ) = \mu_k(B) = \mu_k(\overline{B})$ for all $k \in \mathbb{N}$ since $\mu(\partial B) = 0$, it follows

$$\liminf_{k\to+\infty}\mu_k(B) = \limsup_{k\to+\infty}\mu_k(B).$$

Thus $\lim_{k\to+\infty}\mu_k(B)$ exists, and

$$\lim_{k\to+\infty}\mu_k(B) = \mu(B),$$

as required.

Finally assume that (iii) holds. Fix $\epsilon > 0$ and $f \in \mathcal{C}_c^+(\mathbb{R}^n)$. Let $R > 0$ be such that $\operatorname{supp}(f) \subseteq B(0,R)$ and $\mu(\partial B(0,R)) = 0$. Choose a partition

$$0 := t_0 < t_1 < \cdots < t_N = 2\|f\|_{L^\infty(\mathbb{R}^n)}$$

of $[0, 2\|f\|_{L^\infty(\mathbb{R}^n)}]$ such that $0 < t_i - t_{i-1} < \epsilon$, and $\mu(f^{-1}\{t_i\}) = 0$ for each $i = 1, \dots, N$. Put $B_i := f^{-1}((t_{i-1}, t_i])$, $i = 2, \dots, N$. Then $\mu(\partial B_i) = 0$ for each $i \geq 2$. Now

$$\begin{aligned}
\sum_{i=2}^N t_{i-1}\mu_k(B_i) = \sum_{i=2}^N t_{i-1} \int_{B_i} d\mu_k &\leq \sum_{i=2}^N \int_{B_i} f\, d\mu_k \\
&\leq \int_{\mathbb{R}^n} f\, d\mu_k \\
&\leq \sum_{i=2}^N t_i \mu_k(B_i) + t_1 \mu_k(B(0,R)),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=2}^N t_{i-1}\mu(B_i) = \sum_{i=2}^N t_{i-1} \int_{B_i} d\mu &\leq \sum_{i=2}^N \int_{B_i} f\, d\mu \\
&\leq \int_{\mathbb{R}^n} f\, d\mu \\
&\leq \sum_{i=2}^N t_i \mu(B_i) + t_1 \mu(B(0,R)).
\end{aligned}$$

Thus (iii) implies

$$\begin{aligned}
&\limsup_{k\to+\infty} \left| \int_{\mathbb{R}^n} f\, d\mu_k - \int_{\mathbb{R}^n} f\, d\mu \right| \\
&\quad\leq \limsup_{k\to+\infty} \left| \left\{ \sum_{i=2}^N t_i\mu_k(B_i) + t_1\mu_k(B(0,R)) \right\} - \sum_{i=2}^N t_{i-1}\mu(B_i) \right| \\
&\quad\leq \limsup_{k\to+\infty} \sum_{i=2}^N |t_i\mu_k(B_i) - t_{i-1}\mu(B_i)| + \limsup_{k\to+\infty} t_1\mu_k(B(0,R)) \\
&\quad= \sum_{i=2}^N |t_i - t_{i-1}|\mu(B_i) + t_1\mu(B(0,R)) \\
&\quad\leq 2\epsilon\mu(B(0,R)).
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, taking the limit at $\epsilon \to 0$ shows that

$$\limsup_{k\to+\infty} \left| \int_{\mathbb{R}^n} f\, d\mu_k - \int_{\mathbb{R}^n} f\, d\mu \right| = 0,$$

and hence

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} f\, d\mu_k = \int_{\mathbb{R}^n} f\, d\mu.$$

The proof is complete. $\square$

t1.9-2 **Theorem 1.1.2** (Weak Compactness for Measures). *Let $\{\mu_k\}_{k=1}^{+\infty}$ be a sequence of Radon measures on $\mathbb{R}^n$ satisfying*

$$\sup_{k\in\mathbb{N}} \mu_k(K) < +\infty$$

for each compact set $K \subseteq \mathbb{R}^n$. Then there exists a subsequence $\{\mu_{k_j}\}_{j=1}^{+\infty}$ and a Radon measure μ on $\mathbb{R}^n$ such that

$$\mu_{k_j} \rightharpoonup \mu \quad \textit{as } j \to +\infty.$$

Proof.

(i). Assume first that

$$\sup_{k\in\mathbb{N}} \mu_k(\mathbb{R}^n) < +\infty. \tag{1.1.1}$$ {eq:1.9-1

(ii). Let $\{f_k\}_{k=1}^{+\infty}$ be a countable dense subset of $\mathcal{C}_c(\mathbb{R}^n)$. Note that (1.1.1) implies that the sequence $\{\int_{\mathbb{R}^n} f_1 \, d\mu_j\}_{j=1}^{+\infty}$ is bounded, for

$$\left| \int_{\mathbb{R}^n} f_1 \, d\mu_j \right| \leq \int_{\mathbb{R}^n} |f_1| \, d\mu_j \leq \max_{x\in \operatorname{supp}(f)} |f(x)| \mu_j(\mathbb{R}^n) < +\infty.$$

Thus we may find a subsequence $\{\mu_j^1\}_{j=1}^{+\infty}$ and $a_1 \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} f_1 \, d\mu_j^1 \to a_1 \quad \text{as} \quad j \to +\infty.$$

Continuing, we find subsequences $\{\mu_j^k\}_{j=1}^{+\infty}$ of $\{\mu_j^{k-1}\}_{j=1}^{+\infty}$ and numbers $a_k \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} f_k \, d\mu_j^k \to a_k \quad \text{as} \quad j \to +\infty$$

for each $k \in \mathbb{N}$. Set $\nu_j := \mu_j^j$. Then

$$\int_{\mathbb{R}^n} f_k \, d\nu_j \to a_k \quad \text{as} \quad j \to +\infty$$

for all $k \in \mathbb{N}$, for if $j \geq k$, then $\nu_j = \mu_j^j \in \{\mu_j^k\}_{j=1}^{+\infty}$. Define $L(f_k) := a_k$, and note that L is linear and

$$|L(f_k)| \leq M \|f_k\|_{L^\infty(\mathbb{R}^n)}$$

by (1.1.1), where

$$M := \sup_{k\in\mathbb{N}} \mu_k(\mathbb{R}^n).$$

By the Hahn–Banach Theorem, L may be uniquely extended to a bounded linear functional $\overline{L}$ defined on all of $\mathcal{C}_c(\mathbb{R}^n)$. Then, by the Riesz Representation Theorem, there exists a unique Radon measure μ on $\mathbb{R}^n$ such that

$$\overline{L}(f) = \int_{\mathbb{R}^n} f \, d\mu$$

for all $f \in \mathcal{C}_c(\mathbb{R}^n)$.

(iii). Choose any $f \in \mathcal{C}_c(\mathbb{R}^n)$. Since $\{f_k\}_{k=1}^{+\infty}$ is dense in $\mathcal{C}_c(\mathbb{R}^n)$, there exists a subsequence $\{f_{k_i}\}_{i=1}^{+\infty}$ such that $f_i \to f$ uniformly. Fix $\epsilon > 0$ and then choose $i \in \mathbb{N}$ so large that

$$\|f_{k_i} - f\|_{L^\infty(\mathbb{R}^n)} < \frac{\epsilon}{4M}. \tag{1.1.2}$$ {eq:1.9-2

Next choose $J \in \mathbb{N}$ so that for all $j > J$,

$$\left| \int_{\mathbb{R}^n} f_{k_i} \, d\nu_j - \int_{\mathbb{R}^n} f_{k_i} \, d\mu \right| < \frac{\epsilon}{2}.$$

Then for any $j > J$, we have by (eq:1.9-2 1.1.2) and the Principle of Uniform Boundedness

$$\begin{aligned}\left| \int_{\mathbb{R}^n} f \, d\nu_j - \int_{\mathbb{R}^n} f \, d\mu \right| &\leq \left| \int_{\mathbb{R}^n} f - f_{k_i} \, d\nu_j \right| + \left| \int_{\mathbb{R}^n} f_{k_i} \, d\nu_j - \int_{\mathbb{R}^n} f_{k_i} \, d\mu \right| + \\ &\qquad \left| \int_{\mathbb{R}^n} f_{k_i} - f \, d\mu \right| \\ &\leq \frac{\epsilon}{2} + \|f - f_{k_i}\|_{L^\infty(\mathbb{R}^n)} \nu_j(\mathbb{R}^n) + \|f - f_{k_i}\|_{L^\infty(\mathbb{R}^n)} \mu(\mathbb{R}^n) \\ &< \epsilon,\end{aligned}$$

as required.

(iv). In the general case that (eq:1.9-1 1.1.1) fails to hold, but

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty$$

for each compact set $K \subseteq \mathbb{R}^n$, we apply the above argument to the measures

$$\mu_k^l := \mu_k \llcorner \overline{B(0,l)}, \quad k, l = 1, 2, \ldots,$$

and use a diagonalization argument. The proof is complete. $\square$

For the remainder of this section, we assume that

(i) $U \subseteq \mathbb{R}^n$ is open;
(ii) $1 \leq p < +\infty$.

Definition (Weak Convergence in $L^p(U)$)**.** *A sequence $\{f_k\}_{k=1}^{+\infty} \subset L^p(U)$ is said to converge weakly to $f \in L^p(U)$, written*

$$f_k \rightharpoonup f \quad \text{in } L^p(U),$$

if

$$\lim_{k \to +\infty} \int_U f_k g \, d\mathcal{L}^n = \int_U f g \, d\mathcal{L}^n$$

for each $g \in L^q(U)$, where p and q are conjugate exponents, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < q \leq +\infty$.

t1.9-3 **Theorem 1.1.3** (Weak Compactness in L^p)**.** *Suppose that $1 < p < +\infty$. Let $\{f_k\}_{k=1}^{+\infty} \subseteq L^p(U)$ satisfying*

$$\sup_{k \in \mathbb{N}} \|f_k\|_{L^p(U)} < +\infty.$$

Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{+\infty}$ of $\{f_k\}_{k=1}^{+\infty}$ and a function $f \in L^p(U)$ such that

$$f_{k_j} \rightharpoonup f \quad \text{in } L^p(U) \quad \text{as } j \to +\infty.$$

Remark. *This assertion is in general false for $p = 1$. The key property here is reflexivity. Recall that $L^p(U)$ is reflexive if and only if $1 < p < +\infty$.*

Definition. *We denote by*

$$\nu := \mu \llcorner f$$

the signed measure with density f with respect to μ, that is, the signed measure

$$\nu(K) = \int_K f \, d\mu,$$

provided that this holds for all compact sets $K \subseteq \mathbb{R}^n$.

Proof.
(i). If $U \neq \mathbb{R}^n$, we extend each function f_k to $\mathbb{R}^n$ by setting $f_k = 0$ on $\mathbb{R}^n \setminus U$. This done, we may assume that $U = \mathbb{R}^n$. We may also assume that

$$f_k \geq 0 \quad \mathcal{L}^n - \text{a.e.},$$

for otherwise we could apply the following analysis to f_k^+ and f_k^-.
(ii). Define the Radon measures

$$\mu_k := \mathcal{L}^n \llcorner f_k, \quad k \in \mathbb{N}.$$

Then for each compact set $K \subseteq \mathbb{R}^n$, by Hölder's inequality, we have

$$\mu_k(K) = \int_K f_k \, d\mathcal{L}^n \leq \|f_k\|_{L^p(K)} \cdot \mathcal{L}^n(K)^{\frac{p-1}{p}} < +\infty,$$

and thus

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty.$$

Therefore, we may apply Theorem t1.9-2 (1.1.2) to obtain a Radon measure μ on $\mathbb{R}^n$ and a subsequence

$$\mu_{k_j} \rightharpoonup \mu.$$

(iii). We now show that $\mu << \mathcal{L}^n$. Let $A \subseteq \mathbb{R}^n$ be bounded with $\mathcal{L}^n(A) = 0$. Fix $\epsilon > 0$ and choose an open bounded set $V \supseteq A$ such that $\mathcal{L}^n(V) < \epsilon$. Then by Theorem t1.9-1 (1.1.1) and Hölder's inequality,

$$\begin{aligned}
\mu(A) \leq \mu(V) &\leq \liminf_{j \to +\infty} \mu_{k_j}(V) = \liminf_{j \to +\infty} \int_V f_{k_j} \, d\mathcal{L}^n \\
&\leq \liminf_{j \to +\infty} \|f_{k_j}\| L^p(V) \cdot \mathcal{L}^n(V)^{\frac{p-1}{p}} \\
&\leq C \epsilon^{\frac{p-1}{p}}.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary and $\frac{p-1}{p} > 0$, $\mu(A) = 0$, as required. Therefore $\mu << \mathcal{L}^n$.
(iv). By the Radon–Nikodym Theorem, there exists $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\mu(A) = \int_A f \, d\mathcal{L}^n$$

for every Borel set $A \subseteq \mathbb{R}^n$.
(v). We prove that $f \in L^p(\mathbb{R}^n)$. Let $\phi \in \mathcal{C}_c(\mathbb{R}^n)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} f\phi \, d\mathcal{L}^n = \int_{\mathbb{R}^n} \phi \, d\mu &= \lim_{j \to +\infty} \int_{\mathbb{R}^n} \phi \, d\mu_{k_j} \\
&= \lim_{j \to +\infty} \int_{\mathbb{R}^n} \phi f_{k_j} d\mathcal{L}^n
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{k\in\mathbb{N}} \|f_{k_j}\|_{L^p}(\mathbb{R}^n)\|\phi\|_{L^q(\mathbb{R}^n)} \\
&\leq C\|\phi\|_{L^q(\mathbb{R}^n)}.
\end{aligned}
$$

Thus

$$
\|f\|_{L^p(\mathbb{R}^n)} = \sup_{\substack{\phi\in\mathcal{C}_c(\mathbb{R}^n)\\ \|\phi\|_{L^q(\mathbb{R}^n)=1}}} \left| \int_{\mathbb{R}^n} f\phi \, d\mathcal{L}^n \right| \leq C < +\infty,
$$

and we see that $f \in L^p(\mathbb{R}^n)$.

(vi). Finally, we show that $f_{k_j} \rightharpoonup f$ in $L^p(\mathbb{R}^n)$. Fix $\epsilon > 0$. By the above,

$$
\int_{\mathbb{R}^n} f_{k_j}\phi \, d\mathcal{L}^n \to \int_{\mathbb{R}^n} f\phi \, d\mathcal{L}^n
$$

as $j \to +\infty$ for all $\phi \in \mathcal{C}_c(\mathbb{R}^n)$. Thus we may choose $J \in \mathbb{N}$ so large so that for all $j > J$,

$$
\left| \int_{\mathbb{R}^n} f_{k_j}\phi - f\phi \, d\mathcal{L}^n \right| < \epsilon \tag{1.1.3}
$$

for all $\phi \in \mathcal{C}_c(\mathbb{R}^n)$. Given $g \in L^q(\mathbb{R}^n)$, choose by the density of $\mathcal{C}_c(\mathbb{R}^n)$ in $L^q(\mathbb{R}^n)$ a function $\phi \in \mathcal{C}_c(\mathbb{R}^n)$ such that

$$
\|g - \phi\|_{L^q(\mathbb{R}^n)} < \epsilon.
$$

Then by (1.1.3), Hölder's inequality, and the Principle of Uniform Boundedness, we have for all $j > J$

$$
\begin{aligned}
\left| \int_{\mathbb{R}^n} f_{k_j} g \, d\mathcal{L}^n - \int_{\mathbb{R}^n} fg \, d\mathcal{L}^n \right| &\leq \int_{\mathbb{R}^n} |f_{k_j} g - f_{k_j}\phi| \, d\mathcal{L}^n + \left| \int_{\mathbb{R}^n} f_{k_j}\phi - f\phi \, d\mathcal{L}^n \right| + \\
&\qquad \int_{\mathbb{R}^n} |f\phi - fg| \, d\mathcal{L}^n \\
&\leq \epsilon + \int_{\mathbb{R}^n} |f_{k_j}||g - \phi| \, d\mathcal{L}^n + \int_{\mathbb{R}^n} |f||\phi - g| \, d\mathcal{L}^n \\
&\leq \epsilon + \epsilon\|f_{k_j}\|_{L^p(\mathbb{R}^n)} + \epsilon\|f\|_{L^p(\mathbb{R}^n)} \\
&\leq (2C+1)\epsilon.
\end{aligned}
$$

The proof is complete. □

2. HAUSDORFF MEASURE

2.1. Definitions and Elementary Properties; Hausdorff Dimension.

Definition ($\mathcal{H}^s_\delta$). *Let $A \subseteq \mathbb{R}^n$, $0 \leq s < +\infty$, $0 < \delta \leq +\infty$. We define*

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\operatorname{diam} C_j)^s : A \subseteq \bigcup_{j=1}^{+\infty} C_j, \operatorname{diam} C_j \leq \delta \right\},$$

where

$$\alpha(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma(1 + \frac{s}{2})}$$

denotes the volume of the unit ball in $\mathbb{R}^s$.

Note in the above definition that s need not be an integer.

Definition ($\mathcal{H}^s$, $s-$Dimensional Hausdorff Measure). *Let $A \subseteq \mathbb{R}^n$, $0 \leq s < +\infty$. We define the $s-$dimensional Hausdorff measure $\mathcal{H}^s$ on $\mathbb{R}^n$ by*

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

Note that taking the limit as $\delta \to 0$ coincides with taking the supremum over $\delta > 0$, for, as $\delta \to 0$, we are taking the infimum over smaller and smaller sets. That is, if $\delta_1 < \delta_2$, then there exist coverings $\{C_j\}_{j=1}^{+\infty}$ of A such that $\operatorname{diam} C_j \leq \delta_2$ but $\operatorname{diam} C_j > \delta_1$.

Remark.

(i) Requiring $\delta \to 0$ forces the coverings to "follow the local geometry" of the set A;

(ii) Recall that

$$\mathcal{L}^n(B(x,r)) = \alpha(n) r^n$$

for all balls $B(x,r) \subseteq \mathbb{R}^n$. In fact if $s = k$ is an integer, then $\mathcal{H}^k$ coincides with the ordinary "$k-$dimensional surface area" on nice sets. This is the reason that the normalizing constant $\alpha(s)$ is included in the definition of $\mathcal{H}^s_\delta$.

t2.1-1 **Theorem 2.1.1.** *$\mathcal{H}^s$ is a Borel regular measure, $0 \leq s < +\infty$.*

Remark.

(i) Recall that this means that $\mathcal{H}^s$ is Borel and for each $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mathcal{H}^s(A) = \mathcal{H}^s(B)$.

(ii) $\mathcal{H}^s$ is **not** *a Radon measure if $0 \leq s < n$, since $\mathbb{R}^n$ is not $\sigma-$finite with respect to $\mathcal{H}^s$.*

Proof.

(i). $\mathcal{H}^s_\delta$ is a measure. Choose $\{A_k\}_{k=1}^{+\infty} \subseteq \mathbb{R}^n$ and suppose that $A_k \subseteq \cup_{j=1}^{+\infty} C_j^k$, where $\operatorname{diam} C_j^k \leq \delta$. Then $\{C_j^k\}_{j,k=1}^{+\infty}$ covers $\cup_{k=1}^{+\infty} A_k$. Thus

$$\mathcal{H}^s_\delta\left(\bigcup_{k=1}^{+\infty} A_k\right) \leq \sum_{k=1}^{+\infty} \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\operatorname{diam} C_j^k)^s.$$

Taking infima over all such covers $\{C_j^k\}_{k=1}^{+\infty}$ of A_k, we find

$$\mathcal{H}^s_\delta\left(\bigcup_{k=1}^{+\infty} A_k\right) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s_\delta(A_k),$$

as required.

(ii). $\mathcal{H}^s$ is a measure. Choose $\{A_k\}_{k=1}^{+\infty} \subseteq \mathbb{R}^n$. Since $\mathcal{H}^s(\cup_{k=1}^{+\infty} A_k) = \sup_{\delta>0} \mathcal{H}^s_\delta(\cup_{k=1}^{+\infty} A_k)$, we have

$$\mathcal{H}^s_\delta\left(\bigcup_{k=1}^{+\infty} A_k\right) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s_\delta(A_k) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(A_k).$$

Taking the limit as $\delta \to 0$ on the LHS shows that

$$\mathcal{H}^s\left(\bigcup_{k=1}^{+\infty} A_k\right) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(A_k).$$

(iii). $\mathcal{H}^s$ is a Borel measure. Choose $A, B \subseteq \mathbb{R}^n$ with $\operatorname{dist}(A,B) > 0$. Select $0 < \delta < \frac{1}{4}\operatorname{dist}(A,B)$. Let $A \cup B \subseteq \cup_{k=1}^{+\infty} C_k$ with $\operatorname{diam} C_k \leq \delta$.

Put

$$\mathcal{A} := \{C_j : C_j \cap A \neq \emptyset\}$$

and

$$\mathcal{B} := \{C_j : C_j \cap B \neq \emptyset\}.$$

Then $A \subseteq \cup_{C_j \in \mathcal{A}} C_j$ and $B \subseteq \cup_{C_j \in \mathcal{B}} C_j$, with $C_i \cap C_j = \emptyset$ if $C_i \in \mathcal{A}$, $C_j \in \mathcal{B}$. Thus

$$\begin{aligned}\sum -j = 1^{+\infty} \frac{\alpha(s)}{2^s}(\operatorname{diam} C_j)^s &\geq \sum_{C_j \in \mathcal{A}} \frac{\alpha(s)}{2^s}(\operatorname{diam} C_j)^s + \sum_{C_j \in \mathcal{B}} \frac{\alpha(s)}{2^s}(\operatorname{diam} C_j)^s \\ &\geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B).\end{aligned}$$

Taking the infimum over all such sets $\{C_j\}_{j=1}^{+\infty}$, $0 < \delta < \frac{1}{4}\operatorname{dist}(A,B)$, we find

$$\mathcal{H}^s_\delta(A \cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B).$$

Letting $\delta \to 0$, we obtain

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Consequently

$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$$

for all $A, B \subseteq \mathbb{R}^n$ with $\operatorname{dist}(A,B) > 0$. By Caratheodory's Criterion, $\mathcal{H}^s$ is a Borel measure.

(iv). $\mathcal{H}^s$ is Borel regular. First note that $\operatorname{diam} \overline{C} = \operatorname{diam} C$ for all $C \subseteq \mathbb{R}^n$. Thus

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s}(\operatorname{diam} C_j)^s : A \subseteq \bigcup_{j=1}^{+\infty} C_j, \operatorname{diam} C_j \leq \delta,\ C_j \text{ closed}\right\}.$$

Choose $A \subseteq \mathbb{R}^n$ such that $\mathcal{H}^s(A) < +\infty$. Then $\mathcal{H}^s_\delta(A) < +\infty$ for all $\delta > 0$. For each $k \geq 1$, choose closed sets $\{C_j^k\}_{j=1}^{+\infty}$ so that $\operatorname{diam} C_j^k \leq \frac{1}{k}$, $A \subseteq \cup_{j=1}^{+\infty} C_j^k$, and

$$\sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s}(\operatorname{diam} C_j^k)^s \leq \mathcal{H}^s_{1/k}(A) + \frac{1}{k}.$$

Put $A_k := \cup_{j=1}^{+\infty} C_j^k$ and $B := \cap_{k=1}^{+\infty} A_k$. Then B is Borel. Also $A \subseteq A_k$ for each $k \in \mathbb{N}$, so $A \subseteq B$. Moreover, since $B \subseteq A_k$ for each k,

$$\mathcal{H}^s_{1/k}(B) \leq \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s}(\operatorname{diam} C_j^k)^s \leq \mathcal{H}^s_{1/k}(A) + \frac{1}{k}.$$

Letting $k \to +\infty$, we find

$$\mathcal{H}^s(B) \le \mathcal{H}^s(A).$$

But since $A \subseteq B$, we have by monotonicity

$$\mathcal{H}^s(A) = \mathcal{H}^s(B).$$

The proof is complete. □

t2.1-2 **Theorem 2.1.2** (Elementary Properties of Hausdorff Measure)**.**

(i) $\mathcal{H}^0$ *is counting measure;*
(ii) $\mathcal{H}^1 = \mathcal{L}^1$ *on* $\mathbb{R}$*;*
(iii) $\mathcal{H}^s \equiv 0$ *on* $\mathbb{R}^n$ *for all* $s > n$*;*
(iv) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$ *for all* $\lambda > 0$, $A \subseteq \mathbb{R}^n$*;*
(v) $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$ *for each affine isometry* $L : \mathbb{R}^n \to \mathbb{R}^n$, $A \subseteq \mathbb{R}^n$.

Proof.
(iv). Fix $0 < \delta \le +\infty$, and suppose that $A \subseteq \cup_{j=1}^{+\infty} C_j$, with $\operatorname{diam} C_j \le \delta$. Then $\lambda A \subseteq \cup_{j=1}^{+\infty} \lambda C_j$, and $\operatorname{diam} \lambda C_j = \lambda \operatorname{diam} C_j \le \lambda\delta$. Thus

$$\begin{aligned}\lambda^s \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\operatorname{diam} C_j)^s &= \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\lambda \operatorname{diam} C_j)^s \\ &\ge \mathcal{H}^s_{\lambda\delta}(\lambda A).\end{aligned}$$

Taking the infimum over all such covers $\{C_j\}_{j=1}^{+\infty}$ of A, we deduce

$$\lambda^s \mathcal{H}^s_\delta(A) \ge \mathcal{H}^s_{\lambda\delta}(\lambda A),$$

and taking the limit as $\delta \to 0$ shows

$$\lambda^s \mathcal{H}^s(A) \ge \mathcal{H}^s(\lambda A.)$$

The reverse inequality may be shown similarly.
(v). This follows at once from (iv) along with the translation invariance of $\mathcal{H}^s$.
(i). First note that $\alpha(0) = 1$. Thus obviously $\mathcal{H}^0(\{a\}) = 1$ for all $a \in \mathbb{R}^n$, and (i) follows.
(ii). Choose $A \subseteq \mathbb{R}$ and $\delta > 0$. Then

$$\begin{aligned}\mathcal{L}^1(A) &= \inf\left\{\sum_{j=1}^{+\infty} \operatorname{diam} C_j : A \subseteq \bigcup_{j=1}^{+\infty} C_j\right\} \\ &\le \inf\left\{\sum_{j=1}^{+\infty} \operatorname{diam} C_j : A \subseteq \bigcup_{j=1}^{+\infty} C_j, \operatorname{diam} C_j \le \delta\right\} \\ &= \mathcal{H}^1_\delta(A) \\ &\le \mathcal{H}^1(A).\end{aligned}$$

On the other hand, set $I_k := [k\delta, (k+1)\delta]$, $k \in \mathbb{Z}$. Then $\operatorname{diam}(C_j \cap I_k) \le \delta$, and, since $\cup_{k=1}^{+\infty} C_j \cap I_k = C_j$,

$$\sum_{k=-\infty}^{+\infty} \operatorname{diam}(C_j \cap I_k) \le \operatorname{diam} C_j.$$

Hence,

$$\begin{aligned}\mathcal{L}^1(A) &= \inf\left\{\sum_{j=1}^{+\infty} \operatorname{diam} C_j : A \subseteq \bigcup_{j=1}^{+\infty} C_j\right\}\\ &\geq \inf\left\{\sum_{j=1}^{+\infty}\sum_{k=-\infty}^{+\infty} \operatorname{diam}(C_j \cap I_k) : A \subseteq \bigcup_{j=1}^{+\infty} C_j\right\}\\ &= \mathcal{H}^1_\delta(A).\end{aligned}$$

Therefore $\mathcal{L}^1 = \mathcal{H}^1_\delta$ for all $\delta > 0$, so that taking the supremum over all $\delta > 0$, we have $\mathcal{L}^1 = \mathcal{H}^1$ on $\mathbb{R}$.

(iii). Fix an integer $m \geq 1$. The unit cube $Q(n)$ in $\mathbb{R}^n$ may be decomposed into m^n cubes with side length $\frac{1}{m}$ and diameter $\frac{\sqrt{n}}{m}$. Thus

$$\mathcal{H}^s_{\sqrt{n}/m}(Q(n)) \leq \sum_{j=1}^{m^n} \alpha(s)\left(\frac{\sqrt{n}}{m}\right)^s = \alpha(s) n^{\frac{s}{2}} m^{n-s},$$

and the RHS tends to zero as $m \to +\infty$ if $s > n$. Hence $\mathcal{H}^s(Q(n)) = 0$, so $\mathcal{H}^s \equiv 0$. The proof is complete. □

A convenient way to check that $\mathcal{H}^s$ vanishes on a set $A \subseteq \mathbb{R}^n$ is the following lemma.

l2-1-1 **Lemma 2.1.1.** *If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^s_\delta(A) = 0$ for some $0 < \delta \leq +\infty$, then $\mathcal{H}^s(A) = 0$.*

Proof. The conclusion is obvious if $s = 0$, and so we may assume that $s > 0$.

Fix $\epsilon > 0$. There exist sets $\{C_j\}_{j=1}^{+\infty}$ such that $A \subseteq \cup_{j=1}^{+\infty} C_j$ and

$$\sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s}(\operatorname{diam} C_j)^s \leq \epsilon.$$

In particular for each $j \in \mathbb{N}$,

$$\operatorname{diam} C_j \leq 2\left(\frac{\epsilon}{\alpha(s)}\right)^{\frac{1}{s}} =: \delta(\epsilon).$$

Hence $\mathcal{H}^s_{\delta(\epsilon)} < \epsilon$. But since $\delta(\epsilon) \to 0$ and $\epsilon \to 0$, we have

$$\mathcal{H}^s(A) = 0.$$

The proof is complete. □

We next want to define the *Hausdorff dimension* of a subset of $\mathbb{R}^n$.

l2.1-2 **Lemma 2.1.2.** *Let $A \subseteq \mathbb{R}^n$ and $0 \leq s < t < +\infty$.*

(i) If $\mathcal{H}^s(A) < +\infty$, then $\mathcal{H}^t(A) = 0$;

(ii) If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$.

Proof.

(i). Let $\mathcal{H}^s(A) < +\infty$ and $\delta > 0$. Then there exist sets $\{C_j\}_{j=1}^{+\infty}$ such that $A \subseteq \cup_{j=1}^{+\infty} C_j$, $\operatorname{diam} C_j \leq \delta$, and

$$\sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s}(\operatorname{diam} C_j)^s \leq \mathcal{H}^s_\delta(A) + 1 \leq \mathcal{H}^s(A) + 1.$$

Then

$$\begin{aligned}
\mathcal{H}^t_\delta(A) &\le \sum_{j=1}^{+\infty} \frac{\alpha(t)}{2^t}(\operatorname{diam} C_j)^t \\
&= \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s}(\operatorname{diam} C_j)^s \cdot (\operatorname{diam} C_j)^{t-s} \\
&\le \frac{\alpha(t)}{\alpha(s)} 2^{s-t}\delta^{t-s}(\mathcal{H}^s(A)+1).
\end{aligned}$$

Sending $\delta \to 0$, we conclude that $\mathcal{H}^t(A) = 0$. This proves (i).

(ii). Assertion (ii) follows at once from (i), by contrapositive. The proof is complete. □

Definition (Hausdorff Dimension)**.** *We define the Hausdorff dimension of a set $A \subseteq \mathbb{R}^n$ by*

$$\mathcal{H}_{\dim}(A) := \inf\{0 \le s < +\infty : \mathcal{H}^s(A) = 0.\}$$

Remark. *Observe for any set $A \subseteq \mathbb{R}^n$ that $\mathcal{H}_{\dim}(A) \le n$. Let $s := \mathcal{H}_{\dim}(A)$. Then by the preceding lemma, $\mathcal{H}^t(A) = 0$ for all $t > s$ and $\mathcal{H}^t(A) = +\infty$ for all $t < s$. Moreover, $\mathcal{H}^s(A)$ may be any number between 0 and $+\infty$, inclusive. The point is that $s = \mathcal{H}_{\dim}$ is the only number such that $\mathcal{H}^s(A)$ can be a positive finite number for any $A \subseteq \mathbb{R}^n$.*

Also note that $\mathcal{H}_{\dim}(A)$ need not be an integer. Even if $\mathcal{H}_{\dim}(A) = k$ is an integer and $0 < \mathcal{H}^k(A) < +\infty$, A need not be a "$k-$dimensional surface" in any sense, and may be extremely complicated geometrically. Examples include Cantor–like subsets A of $\mathbb{R}^n$ and other fractals.

2.2. **Isodiametric Inequality; $\mathcal{H}^n = \mathcal{L}^n$.** We want to prove that $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$, where $n \in \mathbb{N}$. Recall that $\mathcal{L}^n$ is defined as the $n-$fold product of one–dimensional Lebesgue measure $\mathcal{L}^1$, so that

$$\mathcal{L}^1(A) := \inf\left\{\sum_{i=1}^{n} \mathcal{L}^n(Q_i) : Q_i \text{ cubes }, A \subseteq \bigcup_{i=1}^{n} Q_i\right\}.$$

On the other hand, $\mathcal{H}^n$ is computed in terms of arbitrary coverings of small diameter.

l2.2-1 **Lemma 2.2.1.** *Let $f : \mathbb{R}^n \to [0, +\infty]$ be L^n-measurable. Then the region "under the graph" of f,*

$$A := \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}, 0 \le y \le f(x)\}$$

is $\mathcal{L}^{n+1}-$measurable.

Proof. Define

$$B := \{x \in \mathbb{R}^n : f(x) = +\infty\}$$

and

$$C := \{x \in \mathbb{R}^n : 0 \le f(x) < +\infty.\}$$

Also define

$$C_{j,k} := \left\{x \in C : \frac{j}{k} \le f(x) < \frac{j+1}{k}\right\}, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{N},$$

so that $C = \cup_{j=0}^{+\infty} C_{j,k}$. Finally, put

$$D_k := \bigcup_{j=0}^{+\infty} \left(C_{j,k} \times \left[0, \frac{j}{k}\right] \right) \cup (B \times [0, +\infty]),$$

$$E_k := \bigcup_{j=0}^{+\infty} \left(C_{j,k} \times \left[0, \frac{j+1}{k}\right] \right) \cup (B \times [0, +\infty]).$$

Clearly D_k and E_k are $\mathcal{L}^{n+1}$ measurable, and we have for each $k \in \mathbb{N}$ $D_k \subseteq A \subseteq E_k$. Write $D := \cup_{k=1}^{+\infty} D_k$ and $E := \cap_{k=1}^{+\infty} E_k$. Then also $D \subseteq A \subseteq E$, with D and E both $\mathcal{L}^{n+1}-$measurable. Now for any $\mathcal{L}^{n+1}-$measurable set F with $\mathcal{L}^{n+1}(F) < +\infty$,

$$\mathcal{L}^{n+1}((E \setminus D) \cap F) \leq \mathcal{L}^{n+1}((E_k \setminus D_k) \cap F) \leq \frac{1}{k} \mathcal{L}^n(F),$$

and the RHS tends to zero as $k \to +\infty$. Thus $\mathcal{L}^{n+1}((E \setminus D) \cap F) = 0$, and, because F was arbitrary, $\mathcal{L}^{n+1}(E \setminus D) = 0$. Hence $\mathcal{L}^{n+1}(A \setminus D) = 0$, and consequently A is $\mathcal{L}^{n+1}-$measurable. □

We now define the process of Steiner symmetrization, which takes a bounded Borel–measurable set $A \subseteq \mathbb{R}^n$ and transforms A into a set $\widetilde{A}$ having the same Lebesgue measure such that $\operatorname{diam}(\widetilde{A}) \leq \operatorname{diam}(A)$.

Fix $a, b \in \mathbb{R}^n$, $\|a\| = 1$. We define

$$L_b^a := \{b + ta : t \in \mathbb{R}\}, \text{ the line through } b \text{ in the direction of } a,$$

and

$$P_a := \{x \in \mathbb{R}^n : x \cdot a = 0\}, \text{ the plane through the origin perpendicular to } a.$$

Definition (Steiner Symmetrization)**.** *Choose $a \in \mathbb{R}^n$ with $\|a\| = 1$, and let $A \subseteq \mathbb{R}^n$. We define the Steiner symmetrization of A with respect to the hyperplane P_a to be the set*

$$S_a(A) := \bigcup_{\substack{b \in P_a \\ A \cap L_b^a \neq \emptyset}} \left\{ b + ta : \|t\| \leq \frac{1}{2} \mathcal{H}^1(A \cap L_b^a) \right\}.$$

Note that the Steiner symmetrization is the union of all line segments $b + ta$ of length less than $\mathcal{H}^1(A \cap L_b^a)$, where b is in the plane through the origin perpendicular to a and there exists $x \in A$ such that $b + ta = x$.

l2.2-2 **Lemma 2.2.2** (Properties of Steiner Symmetrization)**.**

(i) $\operatorname{diam} S_a(A) \leq \operatorname{diam} A$.

(ii) If A is $\mathcal{L}^n-$measurable, then so is $S_a(A)$, and $\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$.

Proof.

(i). Statement (i) is trivial if $\operatorname{diam} A = +\infty$, so we may assume that $\operatorname{diam} A < +\infty$. We may also suppose that A is closed, for

$$\operatorname{diam} A^\circ = \operatorname{diam} A = \operatorname{diam} \overline{A}.$$

Fix $\epsilon > 0$ and choose $x, y \in S_a(A)$ such that

$$\operatorname{diam} S_a(A) \leq \|x - y\| + \epsilon.$$

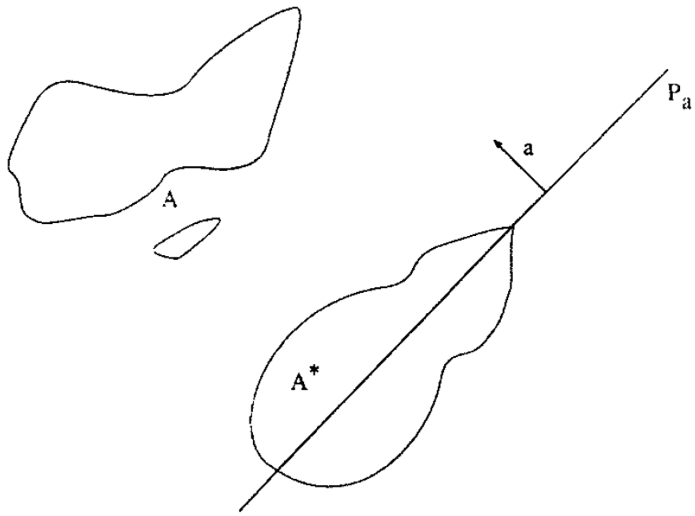

FIGURE 2.2.1. Steiner Symmetrization.

Write $b := x - (x \cdot a)a$ and $c := y - (y \cdot a)a$. Then $b, c \in P_a$. Put

$$\begin{aligned} r &:= \inf\{t : b + ta \in A\}, \\ s &:= \sup\{t : b + ta \in A\}, \\ u &:= \inf\{t : c + ta \in A\}, \\ v &:= \sup\{t : c + ta \in A\}. \end{aligned}$$

Without loss of generality, we may assume that $v - r \geq s - u$. Then

$$\begin{aligned} v - r &\geq \frac{1}{2}(v - r) + \frac{1}{2}(s - u) \\ &= \frac{1}{2}(s - r) + \frac{1}{2}(v - u) \\ &\geq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a) + \frac{1}{2}\mathcal{H}^1(A \cap L_c^a). \end{aligned}$$

Now, $|x \cdot a| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a)$, $|y \cdot a| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a)$, and consequently,

$$v - r \geq |x \cdot a| + |y \cdot a| \geq |x \cdot a - y \cdot a|.$$

Hence,

$$\begin{aligned} (\operatorname{diam} S_a(A) - \epsilon)^2 &\leq \|x - y\|^2 \\ &= \|x\|^2 - 2x \cdot y + \|y\|^2 \\ &= \|b\|^2 + 2(x \cdot a)(b \cdot a) + |x \dot{a}|^2 - 2(b + (x \cdot a)a) \cdot (c + (y \cdot a)a) + \|c\|^2 + \\ &\qquad 2(y \cdot a)(b \cdot a) + |y \cdot a|^2 \\ &= (\|b\|^2 - 2b \cdot c + \|c\|^2) + (|x \cdot a|^2 - 2(x \cdot a)(y \cdot a) + |y \cdot a|^2) + \\ &\qquad 2(x \cdot a)(b \cdot a) - 2(b \cdot a)(y \cdot a) - 2(c \cdot a)(x \cdot a) + 2(y \cdot a)(b \cdot a) \\ &= \|b - c\|^2 + \|x \cdot a - y \cdot a\|^2 \end{aligned}$$

$$\begin{aligned}
&\le \|b-c\|^2 + (v-r)^2 \\
&= \|b\|^2 - 2b\cdot c + \|c\|^2 + v^2 - 2rv + r^2 \\
&= (\|b\|^2 + 2b\cdot ra + \|ra\|^2) - 2(b\cdot c - b\cdot va - c\cdot ra - rv\|a\|^2) + \\
&\qquad (\|c\|^2 + 2c\cdot va + \|va\|^2) \\
&= \|(b+ra) - (c+va)\|^2 \\
&\le (\operatorname{diam} A)^2,
\end{aligned}$$

since $b, c \perp a$ and A is closed, so that $b + ra, c + va \in A$. Thus $\operatorname{diam} S_a(A) - \epsilon \le \operatorname{diam} A$, and since $\epsilon > 0$ was arbitrary, this proves (i).

(ii). Since $\mathcal{L}^n$ is rotation invariant, we may assume that $a = e_n$. Then $P_a = P_{e_n} = \mathbb{R}^{n-1}$. Since $\mathcal{L}^1 = \mathcal{H}^1$ on $\mathbb{R}$, Tonelli's Theorem implies that the map $f : \mathbb{R}^{n-1} \to \mathbb{R}$ defined by $f(b) = \mathcal{H}^1(A \cap L_b^a)$ is $\mathcal{L}^{n-1}-$measurable and $\mathcal{L}^n(A) = \int_{\mathbb{R}^{n-1}} f(b)\, d\mathcal{L}^{n-1}(b)$, for

$$\int_{\mathbb{R}^{n-1}} f(b)\, d\mathcal{L}^{n-1}(b) = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(A \cap L_b^a)\, d\mathcal{L}^{n-1}(b) = \mathcal{L}^n(A).$$

Therefore

$$S_a(A) = \left\{(b, y) : 0 \le |y| \le \frac{f(b)}{2}\right\} \setminus \{(b, 0) : L_b^a \cap A = \emptyset\}$$

is $\mathcal{L}^n-$measurable by Lemma (2.2.1), and

$$\begin{aligned}
\mathcal{L}^n(S_a(A)) = \int_{\mathbb{R}}^n \mathbb{1}_{S_a(A)}\, d\mathcal{L}^n &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \mathbb{1}_{S_a(A)}\, d\mathcal{L}^1\, d\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (\mathbb{1}_{S_a(A)})_{(e_1,\ldots,e_{n-1})}(y)\, d\mathcal{L}^1(y)\, d\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} \int_{-f(b)/2}^{f(b)/2} d\mathcal{L}^1\, d\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} f(b)\, d\mathcal{L}^{n-1}(b) = \mathcal{L}^n(A).
\end{aligned}$$

The proof is complete. □

Remark. *In proving $\mathcal{H}^n = \mathcal{L}^n$ below, notice that we use only statement (ii) above in the special case that a is a standard coordinate vector. Since $\mathcal{H}^n$ is obviously rotation invariant, we in fact prove that $\mathcal{L}^n$ is rotation invariant also.*

t2.2-1 **Theorem 2.2.1** (Isodiametric Inequality)**.** *For all sets $A \subseteq \mathbb{R}^n$,*

$$\mathcal{L}^n(A) \le \frac{\alpha(n)}{2^n}(\operatorname{diam} A)^n.$$

Remark.

(i) Geometrically, the isodiametric inequality says that of all sets of fixed diameter in $\mathbb{R}^n$, the $n-$sphere has greatest volume.

(ii) This inequality is particularly interesting because it is not necessarily the case that A is contained in a ball of diameter $\operatorname{diam} A$, for in $\mathbb{R}^2$ consider the case of an equilateral triangle

with side length 1. The smallest closed ball B which inscribes the triangle has radius $1/\sqrt{3}$, so

$$\operatorname{diam} B = \frac{2}{\sqrt{3}} > 1.$$

Proof. If $\operatorname{diam} A = +\infty$, the inequality is trivial. Therefore we may assume that $\operatorname{diam} A < +\infty$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Define $A_1 := S_{e_1}(A)$, $A_2 := S_{e_2}(A_1), \ldots,$ $A_n := S_{e_n}(A_{n-1})$. Write $A^* := A_n$.

(i). We first show that A^* is symmetric with respect to the origin. We use induction. Clearly A_1 is symmetric with respect to P_{e_1}. Let k be an integer such that $1 \leq k < n$ and suppose that A_k is symmetric with respect to $P_{e_1}, \ldots, P_{e_k}$. Clearly $A_{k+1} = S_{e_{k+1}}(A_k)$ is symmetric with respect to $P_{e_{k+1}}$. Fix $1 \leq j < k$ and let $S_j : \mathbb{R}^n \to \mathbb{R}^n$ be the reflection through P_{e_j}. Let $b \in P_{e_{k+1}}$. Since A_k is symmetric with respect to $P_{e_1}, \ldots, P_{e_k}$ by the induction hypothesis and $1 \leq j \leq k$, we have $S_j(A_k) = A_k$, and so

$$\mathcal{H}^1(A_k \cap L_b^{e_{k+1}}) = \mathcal{H}^1(A_k \cap L_{S_j b}^{e_{k+1}}).$$

Consequently

$$\{t \in \mathbb{R} : b + te_{k+1} \int A_{k+1}\} = \{t \in \mathbb{R} : S_j b + te_{k+1} \in A_{k+1}\}.$$

Thus $S_j(A_{k+1}) = A_{k+1}$, that is, A_{k+1} is symmetric with respect to P_{e_j}. Since j was arbitrary, $A^* = A_n$ is symmetric with respect to $P_{e_1}, \ldots, P_{e_n}$, and so with respect to the origin.

(ii). We show that

$$\mathcal{L}^n(A^*) \leq \frac{\alpha(n)}{2^n}(\operatorname{diam} A^*)^n.$$

Choose $x \in A^*$. Then $-x \in A^*$ by (i), and so $\operatorname{diam} A^* \geq 2|x|$. Thus $A^* \subseteq B(0, \frac{1}{2}\operatorname{diam} A^*)$, and it follows by monotonicity of the Lebesgue measure

$$\mathcal{L}^n(A^*) \leq \mathcal{L}^n\left(B\left(0, \frac{1}{2}\operatorname{diam} A^*\right)\right) = \frac{\alpha(n)}{2^n}(\operatorname{diam} A^*)^2.$$

(iii). We now prove the isodiametric inequality. Note that $\overline{A}$ is $\mathcal{L}^n-$measurable, and thus the above Lemma (2.2.2) implies that

$$\mathcal{L}^n((\overline{A})^*) = \mathcal{L}^n(\overline{A}),$$

as well as

$$\operatorname{diam}(\overline{A})^* \leq \operatorname{diam} \overline{A}.$$

Hence, monotonicity of the Lebesgue measure together with (ii) give

$$\begin{aligned}
\mathcal{L}^n(A) \leq \mathcal{L}^n(\overline{A}) &= \mathcal{L}^n((\overline{A})^*) \\
&\leq \frac{\alpha(n)}{2^n}(\operatorname{diam}(\overline{A})^*)^n \\
&\leq \frac{\alpha(n)}{2^n}(\operatorname{diam}(\overline{A}))^n \\
&= \frac{\alpha(n)}{2^n}(\operatorname{diam} A)^n.
\end{aligned}$$

The proof is complete. ∎

t2.2-2 **Theorem 2.2.2.** *On $\mathbb{R}^n$, $\mathcal{L}^n = \mathcal{H}^n$.*

Proof. (i). We first show that $\mathcal{L}^n(A) \leq \mathcal{H}^n(A)$ for all $A \subseteq \mathbb{R}^n$. Fix $\delta > 0$. Choose sets $\{C_j\}_{j=1}^{+\infty}$ such that $A \subseteq \cup_{j=1}^{+\infty} C_j$ and $\operatorname{diam} C_j \leq \delta$. Then by monotonicity and the Isodiametric Inequality (cf. (2.2.1)),

$$\mathcal{L}^n(A) \leq \sum_{j=1}^{+\infty} \mathcal{L}^n(C_j) \leq \sum_{j=1}^{+\infty} \frac{\alpha(n)}{2^n} (\operatorname{diam} C_j)^n.$$

Taking the infimum of the RHS over all cover countable covers of A with diameter less than δ, we obtain $\mathcal{L}^n(A) \leq H^n_\delta(A)$. Taking the limit as $\delta \to 0$, we have

$$\mathcal{L}^n(A) \leq \mathcal{H}^n_\delta(A) \leq \mathcal{H}^n(A),$$

as required.

(ii). From the definition of $\mathcal{L}^n$ as the $n-$fold product of $\mathcal{L}^1 \times \cdots \times \mathcal{L}^1$, we see that for all $A \subseteq \mathbb{R}^n$ and $\delta > 0$,

$$\mathcal{L}^n(A) = \inf \left\{ \sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) : Q_i \text{ cubes}, A \subseteq \bigcup_{i=1}^{+\infty}, \operatorname{diam} Q_i \leq \delta \right\}.$$

We may consider only cubes parallel to the coordinate axes in $\mathcal{L}^n$.

(iii). We now show that $\mathcal{H}^n$ is absolutely continuous with respect to $\mathcal{L}^n$. Set $C_n := \frac{\alpha(n)}{2^n}$. Then for each cube $Q \subseteq \mathbb{R}^n$,

$$\frac{\alpha(n)}{2^n} (\operatorname{diam} Q)^n = C_n \mathcal{L}^n(Q).$$

Thus for any $A \subseteq \mathbb{R}^n$,

$$\begin{aligned}
\mathcal{H}^n_\delta(A) &= \inf \left\{ \sum_{i=1}^{n} \frac{\alpha(n)}{2^n} (\operatorname{diam} U_i)^n : A \subseteq \bigcup_{i=1}^{+\infty} U_i, \operatorname{diam} U_i \leq \delta \right\} \\
&\leq \inf \left\{ \sum_{i=1}^{+\infty} \frac{\alpha(n)}{2^n} (\operatorname{diam} Q_i)^n : Q_i \text{ cubes }, A \subseteq \bigcup_{i=1}^{+\infty} Q_i, \operatorname{diam} Q_i \leq \delta \right\} \\
&= C_n \mathcal{L}^n(A).
\end{aligned}$$

Taking the supremum over all $\delta > 0$, we've:

$$\mathcal{H}^n(A) \leq C_n \mathcal{L}^n(A).$$

Thus $\mathcal{H}^n(A) = 0$ whenever $\mathcal{L}^n(A) = 0$. This proves (iii).

(iv). We now show that $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$ for all $A \subseteq \mathbb{R}^n$. To this end, fix $\delta > 0$ and $\epsilon > 0$. We may choose cubes $\{Q_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$ such that $A \subseteq \cup_{i=1}^{+\infty} Q_i$, $\operatorname{diam} Q_i \leq \delta$, and

$$\sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) < \mathcal{L}^n(A) + \epsilon.$$

Now for each $i \in \mathbb{N}$ there exist disjoint closed balls $\{B^i_k\}_{k=1}^{+\infty} \subseteq Q_i^\circ$ such that

$$\operatorname{diam} B^i_k \leq \delta$$

and

$$\mathcal{L}^n\left(Q_i \setminus \bigcup_{k=1}^{+\infty} B_k^i\right) = \mathcal{L}^n\left(Q_i^\circ \setminus \bigcup_{k=1}^{+\infty} B_k^i\right) = 0.$$

Since $\mathcal{H}^n, \mathcal{H}^n_\delta$ are absolutely continuous with respect to $\mathcal{L}^n$ by (iii), $\mathcal{H}^n(Q_i \setminus \cup_{k=1}^{+\infty} B_k^i) = \mathcal{H}^n_\delta(Q_i \setminus \cup_{k=1}^{+\infty} B_k^i) = 0$. Therefore $\mathcal{H}^n(Q_i) = \mathcal{H}^n(\cup_{k=1}^{+\infty} B_k^i)$ and $\mathcal{H}^n_\delta(Q_i) = \mathcal{H}^n_\delta(\cup_{k=1}^{+\infty} B_k^i)$, and we have

$$\begin{aligned}
\mathcal{H}^n_\delta(A) &\le \sum_{i=1}^{+\infty} \mathcal{H}^n_\delta(Q_i) = \sum_{i=1}^{+\infty} \mathcal{H}^n_\delta\left(\bigcup_{k=1}^{+\infty} B_k^i\right) \le \sum_{i=1}^{+\infty}\sum_{k=1}^{+\infty} \mathcal{H}^n_\delta(B_k^i) \le \sum_{i=1}^{+\infty}\sum_{k=1}^{+\infty} \mathcal{H}^n(B_k^i) \\
&= \sum_{i=1}^{+\infty}\sum_{k=1}^{+\infty} \frac{\alpha(n)}{2^n}(\operatorname{diam} B_k^i)^n = \sum_{i=1}^{+\infty}\sum_{k=1}^{+\infty} \mathcal{L}^n(B_k^i) = \sum_{i=1}^{+\infty} \mathcal{L}^n\left(\bigcup_{k=1}^{\infty} B_k^i\right) \\
&= \sum_{i=1}^{+\infty}\sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) < \mathcal{L}^n(A) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows $\mathcal{H}^n(A) \le \mathcal{L}^n(A)$. The proof is complete. □

2.3. **Densities.** We first recall the Lebesgue Density Theorem:

Theorem (Lebesgue Density Theorem)**.** *Let $E \subseteq \mathbb{R}^n$ be a Lebesgue measurable set. For any $r > 0$ and $x \in \mathbb{R}^n$, define the approximate Lebesgue density of E in the $r-$neighborhood of x by*

$$d_r(x) := \frac{\mathcal{L}^n(B(x,r) \cap E)}{\alpha(n) r^n}.$$

Further define the Lebesgue density of E at x by

$$d(x) := \lim_{r \to 0} d_r(x).$$

Then

$$d(x) = \lim_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap E)}{\alpha(n) r^n} = \begin{cases} 1, & \text{for } \mathcal{L}^n - \text{a.e. } x \in E, \\ 0, & \text{for } \mathcal{L}^n - \text{a.e. } x \in \mathbb{R}^n \setminus E. \end{cases}$$

Since $\mathcal{H}^n = \mathcal{L}^n$ for $n \in \mathbb{N}$, the above result clearly holds for $\mathcal{H}^n$ as well. We want to develop some analogous results for lower–dimensional Hausdorff measures. Thus we assume throughout this section that $0 < s < n$.

We first establish a theorem that tells us the lower–dimensional Hausdorff density of a set at a.e. point outside the set is zero.

t2.3-1 **Theorem 2.3.1.** *Assume that $E \subseteq \mathbb{R}^n$ with E $\mathcal{H}^s-$measurable and $\mathcal{H}^s(E) < +\infty$. Then*

$$\lim_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} = 0$$

for $\mathcal{H}^s-$a.e. $x \in \mathbb{R}^n \setminus E$.

Proof. Fix $t > 0$ and define

$$A_t := \left\{ x \in \mathbb{R}^n \setminus E : \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} > t \right\}.$$

It suffices to show that $\mathcal{H}^s(A_t) = 0$.

Note that $\mathcal{H}^s \llcorner E$ is a Radon measure, and so, if we fix $\epsilon > 0$, there exists a compact set $K \subseteq E$ such that

$$\mathcal{H}^s(E \setminus K) \leq \epsilon.$$

Set $U := \mathbb{R}^n \setminus K$. Then U is open and $A_t \subseteq U$ because $K \subseteq E$. Fix $\delta > 0$ and consider

$$\mathcal{F} := \left\{ B(x,r) : B(x,r) \subseteq U, 0 < r < \delta, \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} > t \right\}.$$

By the Vitali Covering Lemma, there exists a countable family of balls $\{B(x_i, r_i)\}_{i=1}^{+\infty}$ such that

$$A_t \subseteq \bigcup_{i=1}^{+\infty} B(x_i, 5r_i).$$

Thus by monotonicity

$$\begin{aligned}
\mathcal{H}^s_{10\delta}(A_t) &\leq \mathcal{H}^s_{10\delta}\left(\bigcup_{i=1}^{+\infty} B(x_i, 5r_i) \right) \leq \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (10 r_i)^s \leq \sum_{i=1}^{+\infty} 5^s \alpha(s) r^s \\
&\leq \frac{5^s}{t} \sum_{i=1}^{+\infty} \mathcal{H}^s(B(x_i, r_i) \cap E) \leq \frac{5^s}{t} \mathcal{H}^s(U \cap E) = \frac{5^s}{t} \mathcal{H}^s(E \setminus K) \\
&\leq \frac{5^s}{t} \epsilon.
\end{aligned}$$

Letting $\delta \to 0$, we obtain $\mathcal{H}^s(A_t) \leq \frac{5^s}{t}\epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\mathcal{H}^s(A_t) = 0$ for each $t > 0$. The proof is complete. $\square$

Now we prove that the lower–dimensional Hausdorff density of a set at a.e. point in the set is nonzero. Note that this contrasts with the Lebesgue Density Theorem: the density may not be 1. However, it is bounded below if we replace the limit with limit superior.

t2.3-2 **Theorem 2.3.2.** *Assume that $E \subseteq \mathbb{R}^n$ with $E \mathcal{H}^s-$measurable and $\mathcal{H}^s(E) < +\infty$. Then*

$$\frac{1}{2^s} \leq \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} \leq 1$$

for $\mathcal{H}^s-$a.e. $x \in E$.

Remark. *It is possible to have*

$$\limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} < 1$$

and

$$\liminf_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} = 0$$

for $\mathcal{H}^s-$a.e. $x \in E$, even if $0 < \mathcal{H}^s(E) < +\infty$.

Proof. (i) We first show the upper inequality. Fix $\epsilon > 0$, $t > 1$, and define

$$B_t := \left\{ x \in E : \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} > t \right\}.$$

Since $\mathcal{H}^s \llcorner E$ is Radon, there exists an open set U containing B_t such that

$$\mathcal{H}^s(U \cap E) \leq \mathcal{H}^s(B_t) + \epsilon.$$

Define

$$\mathcal{F} := \left\{ B(x,r) : B(x,r) \subseteq U, 0 < r < \delta, \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} > t \right\}.$$

By a corollary of the Vitali Covering Lemma, there exists a countable family of disjoint balls $\{B(x_i, r_i)\}_{i=1}^{+\infty}$ such that

$$B_t \subseteq \left(\bigcup_{i=1}^{m} B(x_i, r_i) \right) \cup \left(\bigcup_{i=m+1}^{+\infty} B(x_i, 5r_i) \right).$$

Thus

$$\begin{aligned}
\mathcal{H}^s_{10\delta}(B_t) &\leq \mathcal{H}^s_{10\delta}\left(\bigcup_{i=1}^{m} B(x_i, r_i) \right) + \mathcal{H}^s_{10\delta}\left(\bigcup_{i=m+1}^{+\infty} B(x_i, 5r_i) \right) \\
&\leq \sum_{i=1}^{m} \frac{\alpha(s)}{2^s} (2r_i)^s + \sum_{i=m+1}^{+\infty} \frac{\alpha(s)}{2^s} (10 r_i)^s \\
&\leq \sum_{i=1}^{m} \alpha(s) r^s + \sum_{i=m+1}^{+\infty} 5^s \alpha(s) r^s \\
&\leq \frac{1}{t} \sum_{i=1}^{m} \mathcal{H}^s(B(x_i, r_i) \cap E) + \frac{5^s}{t} \sum_{i=m+1}^{+\infty} \mathcal{H}^s(B(x_i, r_i) \cap E) \\
&\leq \frac{1}{t} \mathcal{H}^s(U \cap E) + \frac{5^s}{t} \mathcal{H}^s\left(\bigcup_{i=m+1}^{+\infty} B(x_i, r_i) \cap E \right).
\end{aligned}$$

Note that this holds for each $m = 1, 2, \ldots$. Thus taking the limit as $m \to \infty$ gives

$$\mathcal{H}^s_{10\delta}(B_t) \leq \frac{1}{t} \mathcal{H}^s(U \cap E) \leq \frac{1}{t}(\mathcal{H}^s(B_t) + \epsilon).$$

Letting $\delta \to 0$, we obtain

$$\mathcal{H}^s(B_t) \leq \frac{1}{t}(\mathcal{H}^s(B_t) + \epsilon),$$

and then taking the limit as $\epsilon \to 0$ gives

$$\mathcal{H}^s(B_t) \leq \frac{1}{t} \mathcal{H}^s(B_t).$$

Since $\mathcal{H}^s(B_t) \leq \mathcal{H}^s(E) < +\infty$, this implies that $\mathcal{H}^s(B_t) = 0$ for each $t > 1$, as required.

(ii) We now show that

$$\limsup_{r \to 0} \frac{\mathcal{H}^s_\infty(B(x,r) \cap E)}{\alpha(s) r^s} \geq \frac{1}{2^s}$$

for $\mathcal{H}^s$−a.e. $x \in E$.

For any $\delta > 0$ and $0 < \tau < 1$, denote by $E(\delta, \tau)$ the set of all points $x \in E$ such that

$$\mathcal{H}^s_\delta(C \cap E) \leq \frac{\alpha(s)}{2^s} \tau (\operatorname{diam} C)^s,$$

whenever $C \subseteq \mathbb{R}^n$, $x \in C$, and $\operatorname{diam} C \leq \delta$. Then if $\{C_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$ with $\operatorname{diam} C_i \leq \delta$, $E(\delta, \tau) \subseteq \cup_{i=1}^{+\infty} c_i$, and $C_i \cap E(\delta, \tau) \neq \emptyset$, we have

$$\mathcal{H}^s_\delta(E(\delta, \tau)) \leq \sum_{i=1}^{+\infty} \mathcal{H}^s_\delta(C_i \cap E(\delta, \tau)) \leq \tau \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\operatorname{diam} C_i)^s.$$

Taking the infimum over all such covers $\{C_i\}_{i=1}^{+\infty}$ of $E(\delta, \tau)$, we see that

$$\mathcal{H}^s_\delta(E(\delta, \tau)) \leq \tau \mathcal{H}^s_\delta(E(\delta, \tau)),$$

and so $\mathcal{H}^s_\delta(E(\delta, \tau)) = 0$, since $0 < \tau < 1$ and $\mathcal{H}^s_\delta(E(\delta, \tau)) \leq \mathcal{H}^s_\delta(E) \leq \mathcal{H}^s(E) < +\infty$. In particular,

$$\mathcal{H}^s(E(1-\delta, \delta)) = 0 \tag{2.3.1}$$

for any $0 < \delta < 1$. Now if $x \in E$ and

$$\limsup_{r \to 0} \frac{\mathcal{H}^s_\infty(B(x,r) \cap E)}{\alpha(s) r^s} < \frac{1}{2^s},$$

there exists $\delta > 0$ such that

$$\frac{\mathcal{H}^s_\infty(B(x,r) \cap E)}{\alpha(s) r^s} < \frac{1-\delta}{2^s} \tag{2.3.2}$$

for all $0 < r \leq \delta$. Thus if $x \in C$ and $\operatorname{diam} C \leq \delta$,

$$\begin{aligned} \mathcal{H}^s_\delta(C \cap E) &= \mathcal{H}^s_\infty(C \cap E) \\ &\leq \mathcal{H}^s_\infty(B(x, \operatorname{diam} C) \cap E) \\ &\leq (1-\delta) \frac{\alpha(s)}{2^s} (\operatorname{diam} C)^s, \end{aligned}$$

by (2.3.2). Consequently $x \in E(\delta, 1-\delta)$, and it follows

$$\left\{ x \in E : \limsup_{r \to 0} \frac{\mathcal{H}^s_\infty(B(x,r) \cap E)}{\alpha(s) r^s} < \frac{1}{2^s} \right\} \subseteq \left\{ \bigcup_{k=2}^{+\infty} E\left(\frac{1}{k}, 1 - \frac{1}{k}\right) \right\}.$$

But since the RHS has $\mathcal{H}^s-$measure zero by (2.3.1), this proves (ii).

(iii) Since $\mathcal{H}^s(B(x,r) \cap E) \geq \mathcal{H}^s_\infty(B(x,r) \cap E)$ for any $x \in E$ and $r > 0$, (ii) immediately gives the required lower estimate

$$\limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} \geq \frac{1}{2^s}.$$

The proof is complete. □

2.4. **Hausdorff Measure and Elementary Properties of Functions.** We establish some properties relating the behavior of certain functions and Hausdorff measure.

. . . .

2.4.1. *Hausdorff Measure and Lipschitz Mappings.*

Definition (Lipschitz)**.** *A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is called Lipschitz if there exists a constant $C > 0$ such that*

$$|f(x) - f(y)| \le C|x - y|$$

for all $x, y \in \mathbb{R}^n$.

Definition (Lipschitz Constant)**.** *We define the Lipschitz constant of a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ by*

$$\mathrm{Lip}(f) := \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|}.$$

Note that for any Lipschitz function f,

$$|f(x) - f(y)| \le \mathrm{Lip}(f)|x - y|.$$

t2.4-1 **Theorem 2.4.1.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $A \subseteq \mathbb{R}^n$, $0 \le s < +\infty$. Then*

$$\mathcal{H}^s(f(A)) \le (\mathrm{Lip}(f))^s \mathcal{H}^s(A).$$

Proof. Fix $\delta > 0$ and choose sets $\{C_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$ such that $\operatorname{diam} C_i \le \delta$, $A \subseteq \cup_{i=1}^{+\infty} C_i$. Then

$$\operatorname{diam} f(C_i) \le \mathrm{Lip}(f) \operatorname{diam} C_i \le \delta \, \mathrm{Lip}(f),$$

and $f(A) \subseteq f(\cup_{i=1}^{+\infty} C_i) = \cup_{i=1}^{+\infty} f(C_i)$. Thus

$$\begin{aligned}
\mathcal{H}^s_{\delta \,\mathrm{Lip}(f)}(f(A)) &\le \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\operatorname{diam} f(C_i))^s \\
&\le (\mathrm{Lip}(f))^s \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\operatorname{diam} C_i)^s.
\end{aligned}$$

Taking the infimum over all such sets $\{C_i\}_{i=1}^{+\infty}$ which cover A, we find on the RHS

$$\mathcal{H}^s_{\delta \,\mathrm{Lip}(f)}(f(A)) \le (\mathrm{Lip}(f))^s \mathcal{H}^s_\delta(A).$$

Taking the limit as $\delta \to 0$, we obtain

$$\mathcal{H}^s(f(A)) \le (\mathrm{Lip}(f))^s \mathcal{H}^s(A),$$

as required. The proof is complete. □

c2.4-1 **Corollary 2.4.1.** *Suppose that $n > k$. Let $P : \mathbb{R}^n \to \mathbb{R}^k$ be the usual projection, $A \subseteq \mathbb{R}^n$, $0 \le s < +\infty$. Then*

$$\mathcal{H}^s(P(A)) \le \mathcal{H}^s(A).$$

Proof. Since P is the standard projection map from $\mathbb{R}^n$ to $\mathbb{R}^k$, $\mathrm{Lip}(P) = 1$. Applying the above theorem (cf. (2.4.1)) gives the required estimate. □

. . . .

2.4.2. *Graphs of Lipschitz Functions.*

Definition (Graph). *For $f : \mathbb{R}^n \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$, we define the graph $\Gamma(f; A)$ of f over A by*

$$\Gamma(f; A) := \{(x, f(x)) : x \in A\} \subseteq \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}.$$

t2.4-2 **Theorem 2.4.2.** *Assume that $f : \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{L}^n(A) > 0$.*

(i) Then $\mathcal{H}_{\dim}(\Gamma(f; A)) \geq n$;

(ii) If f is Lipschitz, then $\mathcal{H}_{\dim}(\Gamma(f; A)) = n$.

Remark. *We thus see that the graph of a Lipschitz function f has the expected Hausdorff dimension (think of a continuous function $f : \mathbb{R} \to \mathbb{R}$). We will see from the Area Formula that $\mathcal{H}^s(\Gamma(f; A))$ can be computed according to the usual rules of calculus.*

Proof.

(i). Let $P : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the usual projection. Then by (2.4.1), c2.4-1

$$\mathcal{H}^n(\Gamma(f; A)) \geq \mathcal{H}^n(A) > 0.$$

Thus $\mathcal{H}^n(\Gamma(f; A)) > 0$, so that $\mathcal{H}_{\dim}(\Gamma(f; A)) \geq n$.

(ii). Let Q denote any cube in $\mathbb{R}^n$ of side length 1. Subdivide Q into k^n subcubes $\{Q_1, \ldots, Q_{k^n}\}$ of side length $\frac{1}{k}$. Note that $\operatorname{diam} Q_i = \frac{\sqrt{n}}{k}$ for each $i = 1, \ldots, k^n$. Define

$$a_j^i := \min_{x \in Q_j} f^i(x), \quad b_j^i := \max_{x \in Q_j} f^i(x),$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, k^n$. Since f is Lipschitz,

$$|b_j^i - a_j^i| \leq \operatorname{Lip}(f) \operatorname{diam} Q_j = \operatorname{Lip}(f) \frac{\sqrt{n}}{k}.$$

For each $j = 1, \ldots, k^n$, put

$$C_j := Q_j \times \prod_{i=1}^{m} (a_j^i, b_j^i).$$

Then

$$\Gamma(f; Q_j \cap A) = \{(x, f(x)) : x \in Q_j \cap A\} \subseteq C_j,$$

and $\operatorname{diam} C_j \leq \frac{C}{k}$ for some constant $C > 0$. Since

$$\Gamma(f; A \cap Q) = \Gamma(f; A \cap \cup_{j=1}^{k_n} Q_j) = \bigcup_{j=1}^{k_n} \Gamma(f; A \cap Q_j) \subseteq \bigcup_{j=1}^{j_n} C_j,$$

we have by monotonicity

$$\begin{aligned} \mathcal{H}^n_{C/k}(G(f; A \cap Q)) &\leq \sum_{j=1}^{k_n} \frac{\alpha(n)}{2^n} (\operatorname{diam} C_j)^n \\ &\leq \frac{k^n \alpha(n)}{2^n} \left(\frac{C}{k}\right)^n = \frac{C^n \alpha(n)}{2^n}. \end{aligned}$$

Then upon letting $k \to +\infty$, we find $\mathcal{H}^n(\Gamma(f; A \cap Q)) < +\infty$, and so $\mathcal{H}_{\dim}(\Gamma(f; A \cap Q)) \leq n$. Recall that this estimate is valid for each cube $Q \subseteq \mathbb{R}^n$ of side length 1. Consequently $\mathcal{H}_{\dim}(\Gamma(f; A)) \leq n$. Applying (i), it follows $\mathcal{H}_{\dim}(\Gamma(f; A)) = n$. The proof is complete. □

. . . .

2.4.3. *The Set Where an Integrable Function is Large.* If a function f is locally integrable, we can estimate the Hausdorff measure of the set where f is locally large.

t2.4-3 **Theorem 2.4.3.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let $0 \le s < n$, and define*

$$\Lambda_s := \left\{ x \in \mathbb{R}^n : \limsup_{r\to 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)|\, d\mathcal{L}^n(y) > 0. \right\}$$

Then

$$\mathcal{H}^s(\Lambda_s) = 0.$$

Proof. We may as well assume that $f \in L^1(\mathbb{R}^n)$. By the Lebesgue Differentiation Theorem,

$$\lim_{r\to 0} ⨍_{B(x,r)} |f(y)|\, d\mathcal{L}^n(y) = |f(x)|$$

for $\mathcal{L}^n-$a.e. $x \in \mathbb{R}^n$, and thus

$$\lim_{r\to 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)|\, d\mathcal{L}^n(y) = \lim_{r\to 0} \alpha(n) r^{n-s} ⨍_{B(x,r)} |f(y)|\, d\mathcal{L}^n(y) = \lim_{r\to 0} \alpha(n) r^{n-s} |f(x)| = 0$$

for $\mathcal{L}^n-$a.e. $x \in \mathbb{R}^n$, since $0 \le s < n$. Hence

$$\mathcal{L}^n(\Lambda_s) = 0.$$

Fix $\epsilon > 0$, $\delta > 0$, $\sigma > 0$. Since f is $\mathcal{L}^n-$integrable, there exists $\eta > 0$ such that $\mathcal{L}^n(\Omega) \le \eta$ implies

$$\int_\Omega |f(x)|\, d\mathcal{L}^n(x) < \sigma.$$

Define

$$\Lambda_s^\epsilon := \left\{ x \in \mathbb{R}^n : \limsup_{r\to 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)|\, d\mathcal{L}^n(y) > \epsilon \right\}.$$

By the above analysis,

$$\mathcal{L}^n(\Lambda_s^\epsilon) = 0.$$

Thus there exists an open set $\Omega \subseteq \mathbb{R}^n$ such that $\Lambda_s^\epsilon \subseteq \Omega$ and $\mathcal{L}^n(\Omega) < \eta$. Put

$$\mathcal{F} := \left\{ B(x,r) : x \in \Lambda_s^\epsilon, 0 < r < \delta, B(x,r) \subseteq \Omega, \int_{B(x,r)} |f(y)|\, d\mathcal{L}^n(y) > \epsilon r^s \right\}.$$

By the Vitali Covering Lemma, there exists a countable family $\{B(x_i, r_i)\}_{i=1}^{+\infty}$ of disjoint balls in $\mathcal{F}$ such that

$$\Lambda_s^\epsilon \subseteq \bigcup_{i=1}^{+\infty} B(x_i, 5r_i).$$

We thus compute

$$\begin{aligned} \mathcal{H}^s_{10\delta}(\Lambda_s^\epsilon) &\le \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\operatorname{diam} B(x_i, 5r_i))^s \le \sum_{i=1}^{+\infty} \alpha(s)(5r_i)^s \\ &\le \frac{\alpha(s) 5^s}{\epsilon} \sum_{i=1}^{+\infty} \int_{B(x_i,r_i)} |f(y)|\, d\mathcal{L}^n(y) \\ &\le \frac{\alpha(s) 5^s}{\epsilon} \int_\Omega |f(y)|\, d\mathcal{L}^n(y) \end{aligned}$$

$$\leq \frac{\alpha(s)5^s}{\epsilon}\sigma.$$

Taking the limit as $\delta \to 0$, we have

$$\mathcal{H}^s(\Lambda_s^\epsilon) \leq \frac{\alpha(s)5^s}{\epsilon}\sigma,$$

and then upon sending $\sigma \to 0$ we obtain

$$\mathcal{H}^s(\Lambda_s^\epsilon) = 0.$$

Since $\epsilon > 0$ was arbitrary, it follows

$$\mathcal{H}^s(\Lambda_s) = 0.$$

The proof is complete. □

3. Area and Coarea Formulas

3.1. Lipschitz Functions, Rademacher's Theorem.

Definition (Lipschitz). *Let $A \subseteq \mathbb{R}^n$. A function $f : A \to \mathbb{R}^m$ is called Lipschitz provided that*

$$|f(x) - f(y)| \leq C|x - y| \tag{3.1.1}$$ {eq:3.1-1

for some constant $C > 0$ and all $x, y \in A$. The smallest constant C such that (3.1.1) [eq:3.1-1] *holds for all $x, y \in A$ is denoted*

$$\mathrm{Lip}(f) := \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in A, x \neq y\right\}.$$

Definition (Locally Lipschitz). *A function $f : A \to \mathbb{R}^m$ is called locally Lipschitz if for each compact set $K \subseteq A$, there exists a constant $C_K > 0$ such that*

$$|f(x) - f(y)| \leq C_K|x - y|$$

for all $x, y \in K$.

t3.1-1 **Theorem 3.1.1** (Extension of Lipschitz Functions). *Assume that $A \subseteq \mathbb{R}^n$, and let $f : A \to \mathbb{R}^m$ be Lipschitz. There exists a Lipschitz function $\overline{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that*

(i) $\overline{f} = f$ *on A;*
(ii) $\mathrm{Lip}(\overline{f}) \leq \sqrt{m}\,\mathrm{Lip}(f)$.

Proof.
(i). First assume that $f : A \to \mathbb{R}$. Define

$$\overline{f}(x) := \inf_{x \in A}\left\{f(a) + \mathrm{Lip}(f)|x - a|\right\}.$$

If $b \in A$, then we have $\overline{f}(b) = f(b)$. This follows because if $b \in A$, then

$$\overline{f}(b) \leq f(b) + \mathrm{Lip}(f)|b - b| = f(b).$$

On the other hand, for all $a \in A$, we've:

$$f(a) + \mathrm{Lip}(f)|b - a| \geq f(a) + \frac{f(b) - f(a)}{|b - a|}|b - a| = f(b).$$

Taking the infimum over all $a \in A$ on the LHS thus gives $\overline{f}(b) \geq f(b)$. Now if $x, y \in \mathbb{R}^n$, then

$$\begin{aligned}
\overline{f}(x) &\leq \inf_{a \in A}\left\{f(a) + \mathrm{Lip}(f)(|x - y| + |y - a|)\right\} \\
&= \inf_{a \in A}\left\{f(a) + \mathrm{Lip}(f)|y - a|\right\} + \mathrm{Lip}(f)|x - y| \\
&= \overline{f}(y) + \mathrm{Lip}(f)|x - y|.
\end{aligned}$$

Similarly

$$\overline{f}(y) \leq \overline{f}(x) + \mathrm{Lip}(f)|x - y|.$$

Therefore

$$\frac{|\overline{f}(x) - \overline{f}(y)|}{|x - y|} \leq \mathrm{Lip}(f)$$

for all $x, y \in A$. This proves the result for functions $f : A \to \mathbb{R}$.

(ii). In the general case $f : A \to \mathbb{R}^m$, $f = (f^1, \dots, f^m)$, define $\overline{f} := (\overline{f}^1, \dots, \overline{f}^m)$, where $\overline{f}^i$, $i = 1, \dots, m$, are defined as in (i). Then

$$|\overline{f}(x) - \overline{f}(y)|^2 = \sum_{i=1}^m \left|\overline{f}^i(x) - \overline{f}^i(y)\right|^2 \leq m(\mathrm{Lip}(f))^2|x-y|^2.$$

Taking square roots,

$$\overline{f}(x) - \overline{f}(y) \leq \sqrt{m}\,\mathrm{Lip}(f)|x-y|,$$

as required. The proof is complete. □

Remark. *In fact there exists an extension $\overline{f}$ of f with $\mathrm{Lip}(\overline{f}) = \mathrm{Lip}(f)$. This is Kirszbraun's Theorem.*

We now prove Rademacher's Theorem, which states that a locally Lipschitz function is differentiable $\mathcal{L}^n$−a.e. Note that the inequality

$$|f(x) - f(y)| \leq \mathrm{Lip}(f)|x-y|$$

says nothing about the possibility of locally approximating f by a linear map.

Definition (Differentiable)**.** *The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be differentiable at $x \in \mathbb{R}^n$ if there exists a linear mapping*

$$L : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$\lim_{y\to x} \frac{|f(y) - f(x) - L(x-y)|}{|x-y|} = 0,$$

or, equivalently,

$$f(y) = f(x) + L(x-y) + o(|y-x|), \quad y \to x.$$

Remark.

(i) Note that this is actually the definition of the Fréchet derivative.
(ii) If such a linear mapping L exists, it is unique, and we write

$$Df(x)$$

for L. We call $Df(x)$ the derivative of f at x.

t3.1-2 **Theorem 3.1.2** (Rademacher's Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable $\mathcal{L}^n$−a.e.*

Proof.
(i). We may assume that $m = 1$, for otherwise, repeat the below argument m times. Since differentiability is a local property, we may as well also suppose that f is Lipschitz.
(ii). Fix any $v \in \mathbb{R}^n$ with $|v| = 1$, and for any $x \in \mathbb{R}^n$, define the Gateaux derivative

$$D_v f(x) := \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}$$

at x, provided that this limit exists.
(iii). We show that $D_v f(x)$ exists for $\mathcal{L}^n$−a.e. $x \in \mathbb{R}^n$. Since f is continuous,

$$\overline{D}_v f(x) = \limsup_{t\to 0} \frac{f(x+tv) - f(x)}{t}$$

$$= \lim_{k \to +\infty} \sup_{\substack{0<|t|<\frac{1}{k} \\ t \in \mathbb{Q}}} \frac{f(x+tv)-f(x)}{t}$$

is Borel measurable, as is

$$\underline{D}_v f(x) := \liminf_{t \to 0} \frac{f(x+tv)-f(x)}{t}.$$

Thus

$$\begin{aligned} A_v &:= \{x \in \mathbb{R}^n : D_v f(x) \text{ does not exist}\} \\ &= \left\{x \in \mathbb{R}^n : \underline{D}_v f(x) < \overline{D}_v f(x)\right\}, \end{aligned}$$

being the complement of the set of all points of which the pointwise limit of measurable functions exists, is Borel measurable.

Now, for each $x, v \in \mathbb{R}^n$ with $|v| = 1$, define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) := f(x+tv).$$

Note that for any $t \in \mathbb{R}$,

$$\begin{aligned} |\phi(t) - \phi(s)| &= |f(x+tv) - f(x+sv)| \leq \mathrm{Lip}(f)|(x+tv) - (x+sv)| \\ &= \mathrm{Lip}(f)|t-s|, \end{aligned}$$

so that ϕ is Lipschitz. Therefore ϕ is absolutely continuous, and thus differentiable $\mathcal{L}^1-$a.e. Thus for any line L parallel to v, the set of all points on L such that f is not differentiable has Lebesgue measure zero. That is,

$$\mathcal{H}^1(A_v \cap L) = 0$$

for each line L parallel to v. Thus the Fubini–Tonelli Theorem implies

$$\mathcal{L}^n(A_v) = 0,$$

as required.

(iv). Noting that

$$\frac{\partial}{\partial x_j} f(x) = D_{e_j} f(x) = \lim_{t \to 0} \frac{f(x+te_j)-f(x)}{t}$$

for each $j = 1, \ldots, n$, we have by (iii) that

$$\nabla f(x) = \left(\frac{\partial}{\partial x_1} f(x), \ldots, \frac{\partial}{\partial x_n} f(x) \right)$$

exists for $\mathcal{L}^n-$a.e. $x \in \mathbb{R}^n$.

(v). Next we show that $D_v f(x) = v \cdot \nabla f(x)$ for $\mathcal{L}^n-$a.e. $x \in \mathbb{R}^n$. Let $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} \left[\frac{f(x+tv)-f(x)}{t} \right] \zeta(x)\, dx &= \frac{1}{t} \left[\int_{\mathbb{R}^n} f(x+tv)\zeta(x)\, dx - \int_{\mathbb{R}^n} f(x)\zeta(x)\, dx \right] \\ &= \frac{1}{t} \left[\int_{\mathbb{R}^n} f(x)\zeta(x-tv)\, dx - \int_{\mathbb{R}^n} f(x)\zeta(x)\, dx \right] \\ &= -\int_{\mathbb{R}^n} f(x) \left[\frac{\zeta(x)-\zeta(x-tv)}{t} \right] dx. \end{aligned}$$

This is the integration by parts formula for difference quotients. Let $t = \frac{1}{k}$ for $k = 1, 2, \dots,$ in the above equality and note that

$$\frac{|f(x + \frac{1}{k}v) - f(x)|}{\frac{1}{k}} \leq \operatorname{Lip}(f).$$

Thus, by Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} D_v f(x)\zeta(x)\,dx &\overset{LDC}{=} -\int_{\mathbb{R}^n} f(x) D_v \zeta(x)\,dx \\
&= -\sum_{j=1}^{n} v_i \int_{\mathbb{R}^n} f(x) \frac{\partial}{\partial x_j}\zeta(x)\,dx \\
&= \sum_{j=1}^{n} v_i \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} f(x)\zeta(x)\,dx \\
&= \int_{\mathbb{R}^n} (v \cdot \nabla f(x))\zeta(x)\,dx,
\end{aligned}$$

where we have used integration by parts and the partial derivatives on f are understood in the a.e. sense. Since the above equality holds for every $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have $D_v f = v \cdot \nabla f$ $\mathcal{L}^n$–a.e.

(vi). Choose $\{v_k\}_{k=1}^{+\infty}$ to be a countable, dense subset of $\partial B(0,1)$. Set

$$A_k := \{x \in \mathbb{R}^n : D_{v_k} f(x),\ \nabla f(x) \text{ exist and } D_{v_k} f(x) = v_k \cdot \nabla f(x)\}$$

for each $k \in \mathbb{N}$. Note that by (iii)-(v), $\mathcal{L}^n(\mathbb{R}^n \setminus A_k) = 0$ for each $k \in \mathbb{N}$. Define

$$A := \bigcap_{k=1}^{+\infty} A_k$$

and observe that

$$\mathcal{L}^n(\mathbb{R}^n \setminus A) = \mathcal{L}^n(\mathbb{R}^n \setminus \cap_{k=1}^{+\infty} A_k) = \mathcal{L}^n(\cup_{k=1}^{+\infty}(\mathbb{R}^n \setminus A_k)) = 0.$$

(vii). We now show that f is differentiable at each point $x \in A$. Fix any $x \in A$. Choose $v \in \partial B(0,1)$, $t \in \mathbb{R}$, $t \neq 0$, and write

$$Q(x, v, t) := \frac{f(x + tv) - f(x)}{t} - v \cdot \nabla f(x).$$

Then if $w \in \partial B(0,1)$, we have

$$\begin{aligned}
|Q(x,v,t) - Q(x,w,t)| &= \left| \frac{f(x+tv) - f(x+tw)}{t} - (v - w) \cdot \nabla f(x) \right| \\
&\leq \left| \frac{f(x+tv) - f(x+tw)}{t} \right| + |(v - w) \cdot \nabla f(x)| \\
&\leq \operatorname{Lip}(f)|v - w| + |\nabla f(x)||v - w| \\
&\leq (1 + \sqrt{n}) \operatorname{Lip}(f)|v - w|. \qquad (3.1.2)
\end{aligned}$$

{eq:3.1-2

Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ so large that if $v \in \partial B(0,1)$, then

$$|v - v_k| \leq \frac{\epsilon}{2(1 + \sqrt{n}) \operatorname{Lip}(f)}$$

for some $k = 1, \dots, N$. Note that since $x \in A$,

$$\begin{aligned}\lim_{t\to 0} Q(x, v_k, t) &= \lim_{t\to 0}\left\{\frac{f(x+tv_k)-f(x)}{t} - v_k\cdot\nabla f(x)\right\}\\ &= D_{v_k} f(x) - v_k\cdot\nabla f(x)\\ &= 0\end{aligned}$$

for each $k = 1, \dots, N$. Thus there exists $\delta > 0$ so that for all $0 < |t| < \delta$,

$$|Q(x, v_k, t)| < \frac{\epsilon}{2} \tag{3.1.3}$$

{eq:3.1-3

holds for each $k = 1, \dots, N$. Consequently for each $v \in \partial B(0,1)$ there exists $k \in \{1, \dots, k\}$ such that

$$\begin{aligned}|Q(x,v,t)| &\le |Q(x,v,t) - Q(x,v_k,t)| + |Q(x,v_k,t)|\\ &< (1+\sqrt{n})\operatorname{Lip}(f)|v-v_k| + \frac{\epsilon}{2}\\ &< \epsilon,\end{aligned}$$

by (3.1.2) [eq:3.1-2] and (3.1.3) [eq:3.1-3], provided that $0 < |t| < \delta$. Note that this is the same $\delta > 0$ for all $v \in \partial B(0,1)$.

Now choose any $x, y \in \mathbb{R}^n$, $y \neq x$. Write

$$v := \frac{y-x}{|y-x|},$$

so that $y = x + tv$, where $t := |x-y|$. Then

$$\begin{aligned}|f(y) - f(x) - \nabla f(x)\cdot(y-x)|| &= |f(x+tv) - f(x) - \nabla f(x)\cdot tv|\\ &= |Q(x,t,v)||t|\\ &< \epsilon|t|,\end{aligned}$$

so that

$$f(y) - f(x) - \nabla f(x)\cdot(y-x) = o(t) = o(|x-y|), \quad y \to x.$$

Hence, f is differentiable at x, with

$$Df(x) = \nabla f(x).$$

The proof is complete. □

c3.1-1 **Corollary 3.1.1.**

(i) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz, and

$$\mathcal{Z} := \{x \in \mathbb{R}^n : f(x) = 0\}.$$

Then $Df(x) = 0$ for $\mathcal{L}^n$–a.e. $x \in \mathcal{Z}$.

(ii) Let $f, g := \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz, and

$$Y := \{x \in \mathbb{R}^n : g(f(x)) = x\}.$$

Then

$$Dg(f(x))Df(x) = I$$

for $\mathcal{L}^n$–a.e. $x \in Y$.

Proof.

(i). We may assume that $m = 1$ in (i), otherwise, repeat the following argument m times.

(ii). Choose $x \in \mathcal{Z}$ so that $Df(x)$ exists, and

$$\lim_{r\to 0} \frac{\mathcal{L}^n(\mathcal{Z} \cap B(x,r))}{\mathcal{L}^n(B(x,r))} = 1. \tag{3.1.4}$$

Note that this holds for $\mathcal{L}^n$−a.e. $x \in \mathcal{Z}$. Since $x \in \mathcal{Z}$, it follows

$$f(y) = Df(x)\cdot(y-x) + o(|y-x|). \tag{3.1.5}$$

By contradiction, suppose that $Df(x) = \alpha \neq 0$, and set

$$S := \left\{ v \in \partial B(0,1) : \alpha\cdot v \geq \frac{1}{2}|\alpha| \right\}.$$

Note that S is nonempty, for otherwise $Df(x) = 0$. Now for each $v \in S$ and $t > 0$, set $y := x + tv$ in (3.1.5) to obtain

$$\begin{aligned} f(x+tv) &= \alpha\cdot tv + o(|tv|) \\ &\geq \frac{|\alpha|}{2}t + o(t). \end{aligned}$$

Hence, there exists $\delta > 0$ such that for all $0 < t < \delta$ and all $v \in S$,

$$f(x+tv) > 0.$$

But this contradicts (3.1.4), since for all $0 < r < \delta$, $B(x,r)\cap\mathcal{Z} = \{x\}$. This proves (i).

(iii). We now show (ii). Define

$$\operatorname{dom} Df := \{x \in \mathbb{R}^n : Df(x) \text{ exists}\}$$

and

$$\operatorname{dom} Dg := \{x \in \mathbb{R}^n : Dg(x) \text{ exists}\}.$$

Put

$$X := Y \cap \operatorname{dom} Df \cap f^{-1}(\operatorname{dom} Dg).$$

Then

$$\begin{aligned} Y \setminus X &= Y \cap \left(Y^C \cup (\operatorname{dom} Df)^C \cup (f^{-1}(\operatorname{dom} Dg))^C\right) \\ &= (Y \setminus \operatorname{dom} Df) \cup (Y \setminus f^{-1}(\operatorname{dom} Dg)) \\ &\subseteq (\mathbb{R}^n \setminus \operatorname{dom} Df) \cup g(\mathbb{R}^n \setminus \operatorname{dom} Dg). \end{aligned} \tag{3.1.6}$$

This follows since if $x \in Y \setminus f^{-1}(\operatorname{dom} Dg)$, then $f(x) \in f(Y) \subseteq \mathbb{R}^n$, and $f(x) \notin \operatorname{dom} Dg$, so that

$$f(x) \in \mathbb{R}^n \setminus \operatorname{dom} Dg.$$

Thus

$$x = g(f(x)) \in g(\mathbb{R}^n \setminus \operatorname{dom} Dg.)$$

By Rademacher's Theorem (cf. (3.1.2)),

$$\mathcal{L}^n(\mathbb{R}^n \setminus \operatorname{dom} Df) = 0$$

and

$$\mathcal{L}^n(\mathbb{R}^n \setminus \operatorname{dom} Dg) = 0.$$

Moreover, since g is Lipschitz (cf. (2.4.1)), we have

$$\mathcal{L}^n(g(\mathbb{R}^n \setminus \operatorname{dom} Dg)) \le (\operatorname{Lip}(g))^n \mathcal{L}^n(\mathbb{R}^n \setminus \operatorname{dom} Dg) = 0.$$

Thus, by (3.1.6),

$$\mathcal{L}^n(Y \setminus X) = 0.$$

Now if $x \in X$, $Dg(f(x))$ and $Df(x)$ exist, and so the chain rule implies

$$Dg(f(x))Df(x) = D(g \circ f)(x)$$

exists. Finally, since $(g \circ f)(x) - x = g(f(x)) - x = 0$ on Y, assertion (i) gives

$$Dg(f(x))Df(x) = D(g \circ f)(x) = I$$

$\mathcal{L}^n-$a.e. on Y. The proof is complete. $\square$

3.2. **Linear Maps and Jacobians.** We first review some basic linear algebra. Our goal in this section is to define the Jacobian of a map $f : \mathbb{R}^n \to \mathbb{R}^m$.

. . . .

3.2.1. *Linear Maps.*

Definition (Orthogonal Linear Map)**.** *A linear map $O : \mathbb{R}^n \to \mathbb{R}^m$ is orthogonal if*

$$Ox \cdot Oy = x \cdot y$$

for all $x, y \in \mathbb{R}^n$.

Definition (Symmetric Linear Map)**.** *A linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric if*

$$x \cdot Sy = Sx \cdot y$$

for all $x, y \in \mathbb{R}^n$.

Definition (Diagonal Linear Map)**.** *A linear map $D : \mathbb{R}^n \to \mathbb{R}^n$ is diagonal if there exist $d_1, \dots, d_n \in \mathbb{R}$ such that*

$$Dx = (d_1x_1, \dots, d_nx_n)$$

for all $x \in \mathbb{R}^n$.

Definition (Adjoint)**.** *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The adjoint of A is the linear map $A^* : \mathbb{R}^m \to \mathbb{R}^n$ defined by*

$$x \cdot A^*y = Ax \cdot y$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

Recall that the existence of adjoints in Euclidean space with the Euclidean metric is guaranteed, and, since $\mathbb{R}^n$ is a Hilbert space under the Euclidean metric, the adjoint operator has the above form by the Riesz Representation Theorem.

t3.2-1 **Theorem 3.2.1.**

(i) $A^{**} = A$;
(ii) $(A \circ B)^* = B^* \circ A^*$;
(iii) If $O : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, then $O^* = O^{-1}$;
(iv) If $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric, then $S^* = S$;

(v) *If* $S : \mathbb{R}^n \to \mathbb{R}^n$ *is symmetric, there exists an orthogonal map* $O : \mathbb{R}^n \to \mathbb{R}^n$ *and a diagonal map* $D : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$S = O \circ D \circ O^{-1};$$

(vi) *If* $O : \mathbb{R}^n \to \mathbb{R}^m$ *is orthogonal, then* $n \le m$ *and*

$$O^* \circ O = I \quad \text{on } \mathbb{R}^n,$$
$$O \circ O^* = I \quad \text{on } O(\mathbb{R}^n).$$

Proof.

(i). Since the dot product is symmetric, we have for all $x, y \in \mathbb{R}^n$ that

$$\begin{aligned} x \cdot (A^{**}y) = x \cdot (A^*)^* y = A^* x \cdot y = y \cdot A^* x = Ay \cdot x \\ = x \cdot Ay. \end{aligned}$$

Since this is for all $x \in \mathbb{R}^n$, assertion (i) follows.

(ii). For any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} x \cdot (A \circ B)^* y = (A \circ B)x \cdot y = A(Bx) \cdot y = Bx \cdot A^* y \\ = x \cdot B^*(A^* y). \end{aligned}$$

This is for all $x \in \mathbb{R}^n$, so this proves (ii).

(iii). Let $x, y \in \mathbb{R}^n$. Then

$$x \cdot y = Ox \cdot Oy = x \cdot O^*(Oy),$$

and

$$x \cdot y = O(O^{-1}x) \cdot y = O^{-1}x \cdot O^* y = x \cdot O(O^* y).$$

This shows $O^* = O^{-1}$.

(iv). If $x, y \in \mathbb{R}^n$, then

$$x \cdot Sy = Sx \cdot y = x \cdot S^* y,$$

and since this is for all $x \in \mathbb{R}^n$, assertion (iv) follows. □

t3.2-2 **Theorem 3.2.2** (Polar Decomposition)**.** *Let* $L : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear mapping.*

(i) *If* $n \le m$*, there exists a symmetric map* $S : \mathbb{R}^n \to \mathbb{R}^n$ *and an orthogonal map* $O : \mathbb{R}^n \to \mathbb{R}^m$ *such that*

$$L = O \circ S.$$

(ii) *If* $n \ge m$*, there exists a symmetric map* $S : \mathbb{R}^m \to \mathbb{R}^m$ *and an orthogonal map* $O : \mathbb{R}^m \to \mathbb{R}^n$ *such that*

$$L = S \circ O^*.$$

Proof.

(i). First suppose $n \le m$. Consider the mapping $C := L^* \circ L : \mathbb{R}^n \to \mathbb{R}^n$. Now for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} Cx \cdot y = (L^* \circ L)x \cdot y = L^*(Lx) \cdot y = Lx \cdot Ly = x \cdot L^*(Ly) = x \cdot (L^* \circ L)y \\ = x \cdot Cy, \end{aligned}$$

and also

$$Cx \cdot x = (L^* \circ L)x \cdot x = L^*(Lx) \cdot x = Lx \cdot Lx \ge 0.$$

Thus C is symmetric and positive semidefinite. Hence there exist $\mu_1, \dots, \mu_n \geq 0$ and an orthonormal basis $\{x_k\}_{k=1}^n$ of $\mathbb{R}^n$ such that

$$Cx_k = \mu_k x_k,$$

$k = 1, \dots, n$. Write $\mu_k := \lambda_k^2, \lambda_k \geq 0$, $k = 1, \dots, n$.

(ii). We show that there exists an orthonormal set $\{z_k\}_{k=1}^n$ in $\mathbb{R}^m$ such that

$$Lx_k = \lambda_k z_k,$$

$k = 1, \dots, n$. To see this, if $\lambda_k \neq 0$, define

$$z_k := \frac{1}{\lambda_k} Lx_k.$$

Then if $\lambda_k, \lambda_l \neq 0$,

$$\begin{aligned} z_k \cdot z_l &= \frac{1}{\lambda_k} Lx_k \cdot \frac{1}{\lambda_l} Lx_l = \frac{1}{\lambda_k \lambda_l} Lx_k \cdot Lx_l = \frac{1}{\lambda_k \lambda_l} x_k \cdot L^*(Lx_l) = \frac{1}{\lambda_k \lambda_l} x_k \cdot Cx_l \\ &= \frac{\lambda_l^2}{\lambda_k \lambda_l} x_k \cdot x_l \\ &= \frac{\lambda_l}{\lambda_k} \delta_{kl}, \end{aligned}$$

by (i) and the fact that $\{x_k\}_{k=1}^n$ is an orthonormal set. Thus the set $\{z_k : \lambda_k \neq 0\}$ is orthonormal. If $\lambda_k = 0$, define z_k to be any unit vector such that the set $\{z_k\}_{k=1}^n$ is orthonormal, applying the Gram–Schmidt process if necessary.

(iii). Define $S : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Sx_k := \lambda_k x_k,$$

$k = 1, \dots, n$ and $O : \mathbb{R}^n \to \mathbb{R}^m$ by

$$Ox_k := z_k,$$

$k = 1, \dots, n$. Then

$$(O \circ S)x_k = O(S_k) = O(\lambda_k)x_k = \lambda_k O x_k = \lambda_k z_k = Lx_k,$$

and, since $\{x_k\}_{k=1}^n$ is a basis for $\mathbb{R}^n$,

$$L = O \circ S.$$

Notice that the mapping S is clearly symmetric. Moreover, O is orthogonal because

$$Ox_k \cdot Ox_l = z_k \cdot z_l = \delta_{kl} = x_k \cdot x_l.$$

This proves assertion (i) of the theorem.

(iv). To prove assertion (ii), we apply assertion (i) to L^* and apply (3.2.1) to obtain

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

The proof is complete. □

We now define the Jacobian of a linear map.

Definition (Jacobian)**.** *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.*

(i) If $n \leq m$, write $L = O \circ S$ (cf. (3.2.2)), and we define the Jacobian of L to be

$$[\![L]\!] := |\det S|;$$

(ii) If $n \geq m$, write $L = S \circ O^*$ (cf. (3.2.2)), and we define the Jacobian of L to be

$$[\![L]\!] := |\det S|.$$

Remark.

(i) It will follow from Theorem (3.2.3) below that the definition of $[\![L]\!]$ is independent of the particular choices of O and S.

(ii) Note that if, say, $n \leq m$, then $L = O \circ S$ implies

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

This is the same O and S, and it clearly follows

$$[\![L]\!] = [\![L^*]\!].$$

t3.2-3 **Theorem 3.2.3.**

(i) If $n \leq m$,

$$[\![L]\!]^2 = \det(L^* \circ L);$$

(ii) If $n \geq m$,

$$[\![L]\!]^2 = \det(L \circ L^*).$$

Proof.

(i). Assume that $n \leq m$, and apply Theorem (3.2.2) to write

$$L = O \circ S$$

and

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

Then

$$L^* \circ L = (S \circ O^*) \circ (O \circ S) = S \circ (O^* \circ O) \circ S = S \circ S = S^2$$

(cf. (3.2.1)). Hence,

$$\det(L^* \circ L) = \det(S^2) = (\det S)^2 = [\![L]\!],$$

as required.

(ii). The proof of (ii) is similar. The proof is complete. □

Theorem (3.2.3) provides us with a nice way to compute the Jacobian $[\![L]\!]$ of a linear map. We augment this with the Binet–Cauchy formula below.

Definition ($\Lambda(m,n)$)**.** *If $n \leq m$, we define*

$$\Lambda(m,n) := \{\lambda : \{1,\dots,n\} \to \{1,\dots,m\} : \lambda \textit{ strictly increasing}\}.$$

Note that this is the set of all functions λ that take $\{1,\dots,n\}$ to $\{1,\dots,m\}$ such that $\lambda(k) > \lambda(l)$ if $k > l$, $k,l \in \{1,\dots,n\}$.

Definition (P_λ)**.** *If $n \leq m$, for each $\lambda \in \Lambda(m,n)$, we define $P_\lambda : \mathbb{R}^m \to \mathbb{R}^n$ by*

$$P_\lambda(x_1,\dots,x_m) := (x_{\lambda(1)},\dots,x_{\lambda(n)}).$$

We may think of P_λ as a mapping that "deletes" points from $(x_1,\dots,x_m)$.

Remark. *For each $\lambda \in \Lambda(m,n)$, there exists an $n-$dimensional subspace*

$$S_\lambda := \operatorname{span}\{e_{\lambda(1)},\dots,e_{\lambda(n)}\} \subseteq \mathbb{R}^m$$

such that P_λ is the projection of $\mathbb{R}^m$ onto S_λ.

t3.2-4 **Theorem 3.2.4** (Binet–Cauchy Formula). *Let $n \leq m$ and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then*
$$[\![L]\!]^2 = \sum_{\lambda \in \Lambda(m,n)} (\det(P_\lambda \circ L))^2.$$

Remark.

(i) To calculate $[\![L]\!]$, we compute the sums of the squares of the determinants of each $n \times n$ submatrix of the $m \times n$ matrix representing L, with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$;

(ii) This is a kind of higher dimensional version of the Pythagorean Theorem.

Proof.

(i). Identifying linear maps with their matrices with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, we write
$$L : +((l_{ij}))_{m \times n}, \quad A := L^* \circ L = ((a_{ij}))_{n \times n};$$
so that
$$a_{ij} = \sum_{k=1}^{m} l_{ki} l_{kj}, \quad i, j = 1, \ldots, n.$$

(ii). Note that
$$[\![L]\!]^2 = \det A = \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)},$$
where Σ denotes the set of all permutations of $\{1, \ldots, n\}$. Thus
$$\begin{aligned}[\![L]\!]^2 &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \sum_{k=1}^{m} l_{ki} l_{k\sigma(i)} \\ &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \sum_{\phi \in \Phi} \prod_{i=1}^{n} l_{\phi(i)i} l_{\phi(i)\sigma(i)},\end{aligned}$$
where Φ denotes the set of all one–to–one mappings of $\{1, \ldots, n\}$ into $\{1, \ldots, m\}$.

(iii). Now for each $\phi \in \Phi$, we can uniquely write $\phi := \lambda \circ \theta$, where $\theta \in \Sigma$ and $\lambda \in \Lambda(m, n)$. Consequently we have
$$\begin{aligned}[\![L]\!]^2 &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \sum_{\lambda \in \Lambda(m,n)} \sum_{\theta \in \Sigma} \prod_{i=1}^{n} l_{\lambda \circ \theta(i), i} l_{\lambda \circ \theta(i), \sigma(i)} \\ &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \sum_{\lambda \in \Lambda(m,n)} \sum_{\theta \in \Sigma} \prod_{i=1}^{n} l_{\lambda(i), \theta^{-1}(i)} l_{\lambda(i), \sigma \circ \theta^{-1}(i)} \\ &= \sum_{\lambda \in \Lambda(m,n)} \sum_{\theta \in \Sigma} \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} l_{\lambda(i), \theta(i)} l_{\lambda(i), \sigma \circ \theta(i)}.\end{aligned}$$
Set $\rho := \sigma \circ \theta$. Then
$$[\![(]\!]L)^2 = \sum_{\lambda \in \Lambda(m,n)} \sum_{\rho \in \Sigma} \sum_{\theta \in \Sigma} \operatorname{sgn}(\theta) \operatorname{sgn}(\rho) \prod_{i=1}^{n} l_{\lambda(i), \theta(i)} l_{\lambda(i), \rho(i)}$$

$$= \sum_{\lambda\in\Lambda(m,n)} \left(\sum_{\theta\in\Sigma} \operatorname{sgn}(\theta) \prod_{i=1}^{n} l_{\lambda(i),\theta(i)} \right)^2$$
$$= \sum_{\lambda\in\Lambda(m,n)} (\det(P_\lambda) \circ L)^2,$$

as required. The proof is complete. □

. . . .

3.2.2. *Jacobians.* Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz mapping. By Rademacher's Theorem (cf. (3.1.2)), f is differentiable $\mathcal{L}^n-$a.e., and therefore $Df(x)$ exists and may be regarded as a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$ for $\mathcal{L}^n-$a.e. $x \in \mathbb{R}^n$. We recall the definition of a gradient matrix.

Definition (Gradient Matrix)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, $f = (f^1, \ldots, f^m)$, we define the gradient matrix*

$$Df(x) := \begin{bmatrix} \frac{\partial}{\partial x_1} f^1(x) & \cdots & \frac{\partial}{\partial x_n} f^1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f^m(x) & \cdots & \frac{\partial}{\partial x_n} f^m(x) \end{bmatrix}.$$

Definition (Jacobian)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, the Jacobian of f is*

$$Jf(x) := \llbracket Df(x) \rrbracket, \quad \mathcal{L}^n - \textit{a.e.}$$

Note that in view of Theorem (3.2.3), we have

$$(Jf(x))^2 = \det(Df(x)^* \circ Df(x)) = \det(Df(x) \circ Df(x)^*).$$

3.3. **The Area Formula.** Throughout this section we assume that

$$n \le m.$$

. . . .

3.3.1. *Preliminaries.*

l3.3-1 **Lemma 3.3.1.** *Suppose that $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, $n \le m$. Then*

$$\mathcal{H}^n(L(A)) = \llbracket L \rrbracket \mathcal{L}^n(A)$$

for all $A \subseteq \mathbb{R}^n$.

Proof.

(i). Write $L := O \circ S$, where $O : \mathbb{R}^n \to \mathbb{R}^m$ is an orthogonal map and $S : \mathbb{R}^n \to \mathbb{R}^n$ a symmetric map (cf (3.2.2)). Recall that $[\![L]\!] = |\det S|$.

(ii). If $[\![L]\!] = 0$, then $\dim S(\mathbb{R}^n) \leq n-1$, and so $\dim L(\mathbb{R}^n) \leq n-1$. Consequently $\mathcal{H}^n(L(A)) = 0$, and the inequality is trivial.

(iii). If $[\![L]\!] > 0$, then

$$\begin{aligned}
\frac{\mathcal{H}^n(L(B(x,r)))}{\mathcal{L}^n(B(x,r))} &= \frac{\mathcal{L}^n(O^* \circ L(B(x,r)))}{\mathcal{L}^n(B(x,r))} \\
&= \frac{\mathcal{L}^n(O^* \circ O \circ S(B(x,r)))}{\mathcal{L}^n(B(x,r))} \\
&= \frac{\mathcal{L}^n(S(B(x,r)))}{\mathcal{L}^n(B(x,r))} \\
&= \frac{\mathcal{L}^n(S(B(0,1)))}{\alpha(n)} \\
&= |\det S| = [\![L]\!].
\end{aligned}$$

(iv). Define $\nu(A) := \mathcal{H}^n(L(A))$ for all $A \subseteq \mathbb{R}^n$. Then ν is a Radon measure, $\nu << \mathcal{L}^n$, and

$$D_{\mathcal{L}^n}\nu(x) = \lim_{r\to 0} \frac{\nu(B(x,r))}{\mathcal{L}^n(B(x,r))} = [\![L]\!]$$

by (iii). Thus for all Borel sets $B \subseteq \mathbb{R}^n$,

$$\mathcal{H}^n(L(B)) = [\![L]\!]\mathcal{L}^n(B).$$

Since ν and $\mathcal{L}^n$ are Radon measures, the same identity holds for all sets $A \subseteq \mathbb{R}^n$. The proof is complete. $\square$

For the remainder of the section we assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz.

l3.3-2 **Lemma 3.3.2.** *Let $A \subseteq \mathbb{R}^n$ be $\mathcal{L}^n-$measurable. Then*

(i) $f(A)$ is $\mathcal{H}^n-$measurable;

(ii) The mapping $y \mapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is $\mathcal{H}^n-$measurable on $\mathbb{R}^m$;

(iii) $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n \leq (\mathrm{Lip}(f))^n \mathcal{L}^n(A)$.

Proof.

(i). We may assume without loss of generality that A is bounded.

(ii). There exist compact sets $K_i \subseteq A$ such that

$$\mathcal{L}^n(K_i) \geq \mathcal{L}^n(A) - \frac{1}{i}, \quad i = 1, \ldots, n.$$

Since $\mathcal{L}^n(A) < +\infty$ by the assumption and A is $\mathcal{L}^n-$measurable, $\mathcal{L}^n(A \setminus K_i) \leq \frac{1}{i}$. Since f is continuous, $f(K_i)$ is compact and thus $\mathcal{H}^n-$measurable. Hence, $f(\cup_{i=1}^{+\infty} K_i) = \cup_{i=1}^{+\infty} f(K_i)$ is $\mathcal{H}^n-$measurable. Moreover

$$\begin{aligned}
\mathcal{H}^n\left(f(A) \setminus f\left(\bigcup_{i=1}^{+\infty} K_i\right)\right) &\leq \mathcal{H}^n\left(f\left(A \setminus \bigcup_{i=1}^{+\infty} K_i\right)\right) \\
&\leq (\mathrm{Lip}(f))^n \mathcal{L}^n\left(A \setminus \bigcup_{i=1}^{+\infty} K_i\right) = 0.
\end{aligned}$$

Thus $f(A)$ is $\mathcal{H}^n-$measurable. This proves (i).

(iii). Put

$$\mathcal{B}_k := \left\{ Q : Q = (a_1, b_1] \times \cdots \times (a_n, b_n], a_i := \frac{c_i}{k}, b_i := \frac{c_i + 1}{k}, c_i \in \mathbb{Z}, i = 1, \dots, n \right\},$$

and notice that

$$\mathbb{R}^n = \bigcup_{Q \in \mathcal{B}_k} Q.$$

Define

$$g_k := \sum_{Q \in \mathcal{B}_k} \mathbb{1}_{f(A \cap Q)},$$

and note that g_k is $\mathcal{H}^n-$measurable by assertion (i). Also $g_k(y)$ gives the number of cubes $Q \in \mathcal{B}_k$ such that $f^{-1}(y) \cap (A \cap Q) \neq \emptyset$. Thus

$$g_k(y) \to \mathcal{H}^0(A \cap f^{-1}(y)) \quad \text{as } k \to +\infty$$

for each $y \in \mathbb{R}^m$, and so $y \mapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is $\mathcal{H}^n-$measurable.

(iv). Note that g_k as defined in (iii) satisfies

$$0 \le g_1 \le g_2 \le \cdots .$$

Thus by the Monotone Convergence Theorem,

$$\begin{aligned}
\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) \, d\mathcal{H}^n(y) &= \int_{\mathbb{R}^m} \lim_{k \to +\infty} g_k(y) \, d\mathcal{H}^n(y) \\
&\overset{MCT}{=} \lim_{k \to +\infty} \int_{\mathbb{R}^m} g_k(y) \, d\mathcal{H}^n(y) \\
&= \lim_{k \to +\infty} \sum_{Q \in \mathcal{B}_k} \mathcal{H}^n(f(A \cap Q)) \\
&\le \limsup_{k \to +\infty} \sum_{Q \in \mathcal{B}_k} (\mathrm{Lip}(f))^n (A \cap Q) \\
&= (\mathrm{Lip}(f))^n \mathcal{L}^n(A),
\end{aligned}$$

as required. The proof is complete. □

13.3-3 **Lemma 3.3.3.** *Let $t > 1$ and define*

$$B := \{x \in \mathbb{R}^n : Df(x) \text{ exists}, \ Jf(x) > 0\}.$$

Then there is a countable collection $\{E_k\}_{k=1}^{+\infty}$ of Borel subsets of $\mathbb{R}^n$ such that

(i) $B = \cup_{k=1}^{+\infty} E_k$;

(ii) $f|_{E_k}$ *is one–to–one,* $k = 1, 2, \dots$;

(iii) For each $k = 1, 2, \dots$, *there exists a symmetric automorphism* $T_k : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\mathrm{Lip}((f|_{E_k}) \circ T_k^{-1}) \le t, \quad \mathrm{Lip}(T_k \circ (f|_{E_k})^{-1}) \le t,$$
$$t^{-n} |\det T_k| \le Jf|_{E_k} \le t^n |\det T_k|.$$

Proof.

(i). Fix $\epsilon > 0$ such that

$$\frac{1}{t} + \epsilon < 1 < t - \epsilon.$$

Let C be a countable dense subset of B and let S be a countable dense subset of the symmetric automorphisms of $\mathbb{R}^n$.

(ii). Then for each $c \in C$ and $T \in S$, and $i = 1, 2, \ldots$, define $E(c, T, i)$ to be the set of all $b \in B \cap B(c, \frac{1}{i})$ satisfying

$$\left(\frac{1}{t} + \epsilon\right) |Tv| \leq |Df(b)v| \leq (t - \epsilon)|Tv| \tag{3.3.1}$$

{eq3.3-1}

for all $v \in \mathbb{R}^n$ and

$$|f(a) - f(b) - Df(b) \cdot (a - b)| \leq \epsilon |T(a - b)| \tag{3.3.2}$$

{eq3.3-2}

for all $a \in B(b, \frac{2}{i})$. Note that $E(c, T, i)$ is a Borel set since Df is Borel measurable. Note that from (3.3.1) and (3.3.2) follows the estimate

$$\frac{1}{t}|T(a - b)| \leq |f(a) - f(b)| \leq t|T(a - b)| \tag{3.3.3}$$

{eq3.3-3}

holding for all $b \in E(c, T, i)$ and $a \in B(b, \frac{2}{i})$.

(iii). We next show that if $b \in E(c, T, i)$, then

$$\left(\frac{1}{t} + \epsilon\right)^n |\det T| \leq Jf(b) \leq (t - \epsilon)^n |\det T|.$$

To see this, first note that Df is a linear map. Thus there exists an orthogonal map $O : \mathbb{R}^n \to \mathbb{R}^m$ and a symmetric map $S : \mathbb{R}^n \to \mathbb{R}^n$ (cf. (3.2.2)) such that $Df = O \circ S$. Then

$$Jf(b) = [\![Df(b)]\!] = |\det S|.$$

By (3.3.1),

$$\left(\frac{1}{t} + \epsilon\right) |Tv| \leq |(O \circ S)v| = |Sv| \leq (t - \epsilon)|Tv|$$

for all $v \in \mathbb{R}^n$, and so

$$\left(\frac{1}{t} + \epsilon\right) |v| \leq |(S \circ T^{-1})v| \leq (t - \epsilon)|v|$$

for all $v \in \mathbb{R}^n$. Thus

$$(S \circ T^{-1})(B(0, 1)) \subset B(0, t - \epsilon),$$

so that

$$|\det(S \circ T^{-1})|\alpha(n) \leq \mathcal{L}^n(B(0, t - \epsilon)) = \alpha(n)(t - \epsilon)^n,$$

and hence

$$|\det S| \leq (t - \epsilon)^n |\det T|.$$

The proof of the reverse inequality follows from the fact that

$$|(S \circ T^{-1})v| \geq \left(\frac{1}{t} + \epsilon\right),$$

and thus

$$B\left(0, \frac{1}{t} + \epsilon\right) \subset (S \circ T^{-1})(B(0, 1)).$$

(iv). Relabel the countable collection $\{E(c, T, i) : c \in C, T \in S, i \in \mathbb{N}\}$ as $\{E_k\}_{k=1}^{+\infty}$. Choose any $b \in B$, write $Df = O \circ S$, and choose $T \in S$ such that

$$\mathrm{Lip}(T \circ S^{-1}) \leq \left(\frac{1}{t} + \epsilon\right)^{-1}, \quad \mathrm{Lip}(S \circ T^{-1}) \leq t - \epsilon.$$

Now choose $i \in \mathbb{N}$ and $c \in C$ such that $|b-c| < \frac{1}{i}$ and

$$|f(a) - f(b) - Df(b) \cdot (a-b)| \leq \frac{\epsilon}{\mathrm{Lip}(T^{-1})}|a-b| \leq \epsilon |T(a-b)|$$

for all $a \in B(b, \frac{2}{i})$. Then by (iii), $b \in E(c, T, i)$. Since this holds for all $b \in B$, this proves assertion (i).

(v). Next choose any set $E_k = E(c, T, i)$. Let $T_k := T$. By (3.3.3),

$$\frac{1}{t}|T_k(a-b)| \leq |f(a) - f(b)| \leq t|T_k(a-b)|$$

for all $b \in E_k$, $a \in B(b, \frac{2}{i})$. Since $E_k \subset B(c, \frac{1}{i}) \subset B(b, \frac{2}{i})$, we have

$$\frac{1}{t}|T_k(a-b)| \leq |f(a) - f(b)| \leq t|T_k(a-b)| \tag{3.3.4}$$

holding for all $a, b \in E_k$. Thus $f|_{E_k}$ is one–to–one.

(vi). Finally notice that (3.3.4) implies

$$\mathrm{Lip}((f|_{E_k}) \circ T_k^{-1}) \leq t, \quad \mathrm{Lip}(T_k \circ (f|_{E_k})^{-1}) \leq t.$$

Thus (iii) provides the esitmate

$$t^{-n}|\det T_k| \leq Jf|_{E_k} \leq t^n |\det T_k|,$$

which proves assertion (iii). The proof is complete. □

. . . .

3.3.2. *Proof of the Area Formula.*

Theorem 3.3.1 (The Area Formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \leq m$. Then for each $\mathcal{L}^n$−measurable subset $A \subseteq \mathbb{R}^n$,*

$$\int_A Jf(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n(y).$$

Proof.

(i). In view of Rademacher's Theorem (cf. (3.1.2)), we may assume that $Df(x)$ and $Jf(x)$ exist for all $x \in A$. We may also assume that $\mathcal{L}^n(A) < +\infty$, for otherwise both sides of the equality are $+\infty$.

(ii). Suppose now that $A \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$. Fix $t > 1$ and choose Borel sets $\{E_k\}_{k=1}^{+\infty}$ as in Lemma (3.3.3). That is,

(1) $B = \cup_{k=1}^{+\infty} E_k$,
(2) $f|_{E_k}$ is one–to–one, $k = 1, 2, \ldots$,
(3) For each $k = 1, 2, \ldots$, there exists a symmetric automorphism $T_k : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathrm{Lip}((f|_{E_k}) \circ T_k^{-1}) \leq t, \quad \mathrm{Lip}(T_k \circ (f|_{E_k})^{-1}) \leq t,$$

and

$$t^{-n}|\det T_k| \leq Jf|_{E_k} \leq t^n |\det T_k|.$$

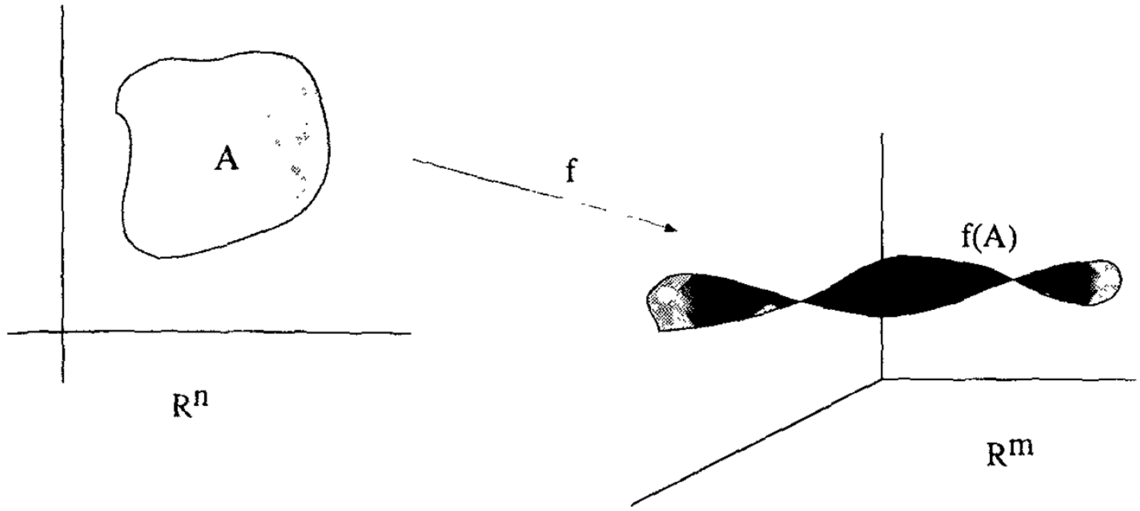

FIGURE 3.3.1. The Area Formula.

Upon passing to the collection $F_k := E_k \setminus (\cup_{i=1}^{k-1} E_k)$ if necessary, we may also suppose that the set $\{E_k\}_{k=1}^{+\infty}$ are disjoint. Define $\mathcal{B}_k$ as in the proof of Lemma (3.3.2), that is,

$$\mathcal{B}_k := \{Q : Q = (a_1, b_1] \times \cdots \times (a_n, b_n], a_i := \frac{c_i}{k}, b_i := \frac{c_i + 1}{k}, c_i \in \mathbb{Z}, i = 1, \ldots, n\}.$$

Set

$$F_j^i := E_j \cap Q_i \cap A, \quad Q_i \in \mathcal{B}_k, \quad j = 1, \ldots, n.$$

Then the sets F_j^i are disjoint because $\{E_k\}_{k=1}^{+\infty}$ is disjoint, and $A = \cup_{i,j=1}^{+\infty} F_j^i$.

(iii). We claim that

$$\lim_{k \to +\infty} \sum_{i,j=1}^{+\infty} \mathcal{H}^n(f(F_j^i)) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n(y).$$

To see this, put

$$g_k := \sum_{i,j=1}^{+\infty} \mathbb{1}_{f(F_j^i)}.$$

Note that $g_k(y)$ is equal to the number of sets $\{F_j^i\}$ such that $F_j^i \cap f^{-1}(y) \neq \emptyset$. Then $g_k(y) \to \mathcal{H}^0(A \cap f^{-1}(y))$ as $k \to +\infty$. Notice that this is also an increasing sequence. Thus by the Monotone Convergence Theorem,

$$\begin{aligned}
\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n(y) &= \int_{\mathbb{R}^m} \lim_{k \to +\infty} g_k(y)\, d\mathcal{H}^n(y) \\
&\overset{MCT}{=} \lim_{k \to +\infty} \int_{\mathbb{R}^m} g_k(y)\, d\mathcal{H}^n(y) \\
&= \lim_{k \to +\infty} \sum_{i,j=1}^{+\infty} \mathcal{H}^n(f(F_j^i)),
\end{aligned}$$

where the last inequality follows from the fact that $\{F^i_j\}$ is disjoint.

(iv). Next note that

$$\mathcal{H}^n(f(F^i_j)) = \mathcal{H}^n(f|_{E_j}(F^i_j)) = \mathcal{H}^n(f|_{E_j} \circ T_j^{-1} \circ T_j(F^i_j)) \leq t^n \mathcal{L}^n(T_j(F^i_j))$$

and

$$\mathcal{L}^n(T_j(F^i_j)) = \mathcal{H}^n(T_j \circ (f|_{E_j})^{-1} \circ f|_{E_j}(F^i_j)) \leq t^n \mathcal{H}^n(f(F^i_j))$$

by Lemma (3.3.3) (cf. (2.4.1)). Thus

$$\begin{aligned} t^{-2n}\mathcal{H}^n(f(F^i_j)) &\leq t^{-n}\mathcal{L}^n(T_j(F^i_j)) \\ &= t^{-n}|\det T_j|\mathcal{L}^n(F^i_j) \\ &\leq \int_{F^i_j} Jf(x)\, d\mathcal{L}^n(x) \\ &\leq t^n|\det T_j|\mathcal{L}^n(F^i_j) \\ &= t^n\mathcal{L}^n(T_j(F^i_j)) \\ &\leq t^{2n}\mathcal{H}^n(f(F^i_j)) \end{aligned}$$

(cf. Lemmas (3.3.1) and (3.3.3)). Now summing on i and j, and recalling that $A = \cup_{i,j=1}^{+\infty} F^i_j$, we have

$$t^{-2n}\sum_{i,j=1}^{+\infty}\mathcal{H}^n(f(F^i_j)) \leq \int_A Jf(x)\, d\mathcal{L}^n(x) \leq t^{2n}\sum_{i,j=1}^{+\infty}\mathcal{H}^n(f(F^i_j)).$$

Letting $k \to +\infty$, we have by (iii) that

$$t^{-2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}(y))\, d\mathcal{H}^n(y) \leq \int_A Jf(x)\, d\mathcal{L}^n(x) \leq t^{2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}(y))\, d\mathcal{H}^n(y).$$

Finally, taking the limit as $t \to 1^+$ shows that

$$\int_A Jf(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}(y))\, d\mathcal{H}^n(y),$$

which completes the proof for the case $A \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$.

(v). Now consider the case $A \subset \{x \in \mathbb{R}^n : Jf(x) = 0\}$. Fix $\epsilon > 0$. We factor $f := p \circ g$, where

$$g : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n, \quad g(x) := (f(x), \epsilon x), \quad x \in \mathbb{R}^n,$$

and

$$p : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m, \quad p(y,z) := y, \quad y \in \mathbb{R}^m,\ z \in \mathbb{R}^n.$$

(vi). We now claim that there exists a constant $C > 0$ such that

$$0 < Jg(x) \leq C\epsilon$$

for all $x \in A$. To prove this claim, write $g = (f^1, \ldots, f^m, \epsilon x_1, \ldots, \epsilon x_m)$. Then

$$Dg(x) = \begin{bmatrix} Df(x) \\ \epsilon I \end{bmatrix}.$$

Since $Jg(x)^2$ equals the sum of squares of the $(n \times n)$ subdeterminants of $Dg(x)$ according to the Binet–Cauchy Formula (cf. (3.2.4)), we see that

$$Jg(x)^2 \geq \epsilon^{2n} > 0.$$

Moreover, since $|Df| \leq \mathrm{Lip}(f) < +\infty$, we may use the Binet–Cauchy formula to also compute

$$Jg(x)^2 = Jf(x)^2 + \{\text{sum of squares of terms each involving at least one } \epsilon\} \leq C\epsilon^2$$

for each $x \in A$.

(vii). Since $p : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a projection, $\mathrm{Lip}(p) \leq 1$, and we can compute using the first case $A \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$

$$\begin{aligned}
\mathcal{H}^n(f(A)) &\leq \mathcal{H}^n(g(A)) \\
&\leq \int_{\mathbb{R}^{n+m}} \mathcal{H}^0(A \cap g^{-1}(y,z))\, d\mathcal{H}^n(y,z) \\
&= \int_A Jg(x)\, d\mathcal{L}^n(x) \\
&\leq C\epsilon \mathcal{L}^n(A).
\end{aligned}$$

Letting $\epsilon \to 0$, we conclude that $\mathcal{H}^n(f(A)) = 0$, and thus

$$\int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n(y) = 0,$$

since $\operatorname{supp} \mathcal{H}^0(A \cap f^{-1}(y)) \subset f(A)$. But then since $Jf(x) = 0$ on A by the assumption, it follows

$$\int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n(y) = 0 = \int_A Jf(x)\, d\mathcal{L}^n(x),$$

as required.

(viii). In the general case, write $A := A_1 \cup A_2$, with $A_1 \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$, $A_2 \subset \{x \in \mathbb{R}^n : Jf(x) = 0\}$, and apply the above arguments. The proof is complete. $\square$

. . . .

3.3.3. *Change of Variables Formula.*

t3.3-2 **Theorem 3.3.2.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \leq m$. Then for each $\mathcal{L}^n$−integrable function $g : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} g(x) Jf(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \left[\sum_{x \in f^{-1}(y)} g(x) \right] d\mathcal{H}^n(y).$$

Proof.

(i). Consider first the case $g \geq 0$. Recall that the sequence $\{s_n\}_{n=1}^{+\infty}$ of simple functions defined by

$$s_j(x) := \sum_{k=0}^{j2^j} \frac{k}{2^j} \mathbb{1}_{g^{-1}[\frac{k}{2^j}, \frac{k+1}{2^j})}(x) + j \mathbb{1}_{g^{-1}[j,+\infty]}(x)$$

satisfies $s_j \to g$ as $j \to +\infty$ and

$$0 \leq s_1 \leq s_2 \leq \cdots .$$

Thus the Monotone Convergence Theorem implies that

$$\int_{\mathbb{R}^n} g(x) Jf(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^n} \lim_{j \to +\infty} s_j(x) Jf(x)\, d\mathcal{L}^n(x)$$

$$
\begin{aligned}
&\overset{MCT}{=} \lim_{j\to+\infty} \int_{\mathbb{R}^n} s_j(x) Jf(x)\, d\mathcal{L}^n(x) \\
&\overset{B.L.}{=} \lim_{j\to+\infty} \sum_{k=1}^{j2^j} \frac{k}{2^j} \int_{g^{-1}[\frac{k}{2^j},\frac{k+1}{2^j})} Jf(x)\, d\mathcal{L}^n(x) \\
&= \lim_{j\to+\infty} \sum_{k=1}^{j2^j} \frac{k}{2^j} \int_{\mathbb{R}^m} \mathcal{H}^0\left(g^{-1}\left[\frac{k}{2^j},\frac{k+1}{2^j}\right) \cap f^{-1}(y)\right)\, d\mathcal{H}^n(y) \\
&\overset{B.L.}{=} \lim_{j\to+\infty} \int_{\mathbb{R}^m} \sum_{k=1}^{+\infty} \frac{k}{2^j} \sum_{x\in f^{-1}(y)} \mathbb{1}_{g^{-1}[\frac{k}{2^j},\frac{k+1}{2^j})}(x)\, d\mathcal{H}^n(y) \\
&\overset{MCT}{=} \int_{\mathbb{R}^m} \sum_{x\in f^{-1}(y)} \lim_{j\to+\infty} \sum_{k=1}^{j2^j} \frac{k}{2^j} \mathbb{1}_{g^{-1}[\frac{k}{2^j},\frac{k+1}{2^j})}(x)\, d\mathcal{H}^n(y) \\
&= \int_{\mathbb{R}^m} \left[\sum_{x\in f^{-1}(y)} g(x)\right]\, d\mathcal{H}^n(y).
\end{aligned}
$$

(ii). Now in the case that g is any $\mathcal{L}^n-$integrable function, write $g = g^+ - g^-$ and apply the above case (i). The proof is complete. □

. . . .

3.3.4. *Applications.*

Example 3.3.1 (Length of a Curve ($n = 1, m \geq 1$)). *Assume that $f : \mathbb{R} \to \mathbb{R}^m$ is Lipschitz and one–to–one. Write*

$$f = (f^1, \dots, f^m), \quad Df = (\dot{f}^1, \dots, \dot{f}^n),$$

so that

$$Jf = |Df| = |\dot{f}|.$$

For any $-\infty < a < b < +\infty$, define the curve

$$C := f([a,b]) \subset \mathbb{R}^m.$$

Then by the Area Formula

$$
\begin{aligned}
\int_a^b |\dot{f}(t)|\, dt &= \int_{[a,b]} Jf(x)\, d\mathcal{L}^1(x) \\
&= \int_{\mathbb{R}^m} \mathcal{H}^0([a,b] \cap f^{-1}(y))\, d\mathcal{L}^1(y) \\
&= \mathcal{H}^1(C).
\end{aligned}
$$

Example 3.3.2 (Surface Area of a Graph ($n \geq 1, m = n+1$)). *Assume that $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and define $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ by*

$$f(x) := (x, g(x)).$$

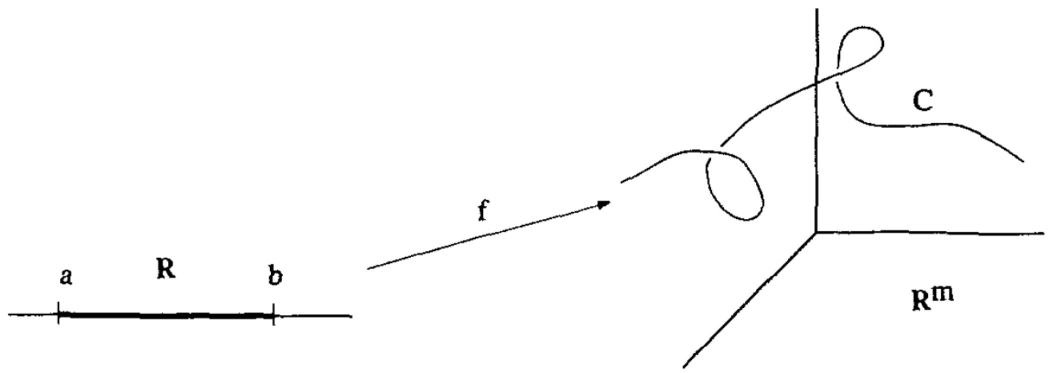

FIGURE 3.3.2. Length of a Curve.

Note that $f = \Gamma(g)$. *Then*

$$Df(x) = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \frac{\partial}{\partial x_1} g(x) & \cdots & \frac{\partial}{\partial x_n} g(x) \end{bmatrix}.$$

By the Binet–Cauchy formula,

$$(Jf)^2 = \textit{sum of squares of } n \times n \textit{ subdeterminants} = 1 + |Dg|^2,$$

so that $Jf = (1 + |Dg|^2)^{1/2}$. *Now for each open set* $\Omega \subset \mathbb{R}^n$, *recall the graph of* g *over* Ω :

$$\Gamma(g, \Omega) = \{(x, f(x)) : x \in \Omega\} \subset \mathbb{R}^{n+1}.$$

Then by the Area Formula

$$\begin{aligned} \int_\Omega (1 + |Dg(x)|^2)^{1/2} \, d\mathcal{L}^n(x) &= \int_\Omega Jf(x) \, d\mathcal{L}^n(x) \\ &= \int_{\mathbb{R}^{n+1}} \mathcal{H}^0(\Omega \cap f^{-1}(y)) \, d\mathcal{H}^n(y) \\ &= \mathcal{H}^n(\Gamma(g, \Omega)). \end{aligned}$$

Example 3.3.3 (Surface Area of a Parametric Hypersurface ($n \geq 1$, $m = n + 1$))**.** *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ *is one–to–one and Lipschitz. Write*

$$f = (f^1, \ldots, f^{n+1})$$

and

$$Df(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f^1(x) & \cdots & \frac{\partial}{\partial x_n} f^1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f^{n+1}(x) & \cdots & \frac{\partial}{\partial x_n} f^{n+1}(x) \end{bmatrix}.$$

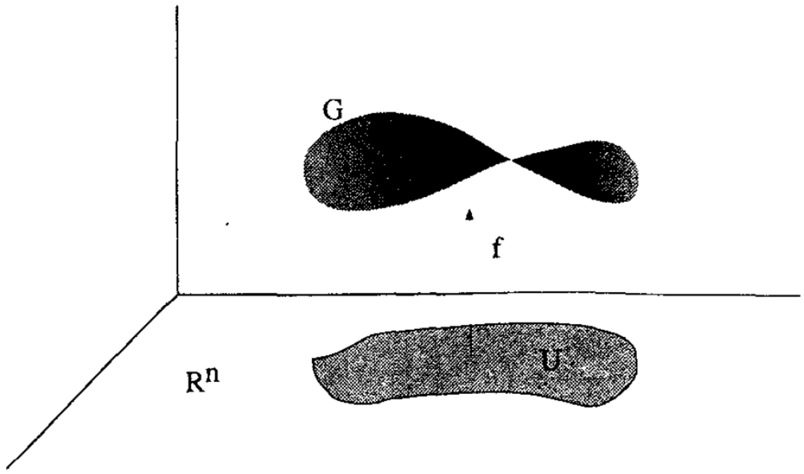

FIGURE 3.3.3. Surface Area of a Graph.

Then by the Binet–Cauchy formula,

$$
\begin{aligned}
(Jf)^2 &= \textit{sum of squares of } n \times n \textit{subdeterminants} \\
&= \sum_{k=1}^{n+1} \left[\frac{\partial(f^1, \ldots, f^{k-1}, f^{k+1}, \ldots, f^{n+1})}{\partial x_1, \ldots, x_n} \right]^2,
\end{aligned}
$$

where

$$
\frac{\partial(f^1, \ldots, f^{k-1}, f^{k+1}, \ldots, f^{n+1})}{\partial x_1, \ldots, x_n}
$$

denotes the Jacobian of the function with gradient matrix

$$
\begin{bmatrix}
\frac{\partial}{\partial x_1} f^1(x) & \cdots & \frac{\partial}{\partial x_n} f^1(x) \\
\vdots & & \vdots \\
\frac{\partial}{\partial x_1} f^{k-1}(x) & \cdots & \frac{\partial}{\partial x_n} f^{k-1}(x) \\
\frac{\partial}{\partial x_1} f^{k+1}(x) & \cdots & \frac{\partial}{\partial x_n} f^{k+1}(x) \\
\vdots & & \vdots \\
\frac{\partial}{\partial x_1} f^{n+1}(x) & \cdots & \frac{\partial}{\partial x_n} f^{n+1}(x)
\end{bmatrix}.
$$

For each open set $\Omega \subset \mathbb{R}^n$, write

$$
S := f(\Omega) \subset \mathbb{R}^{n+1}.
$$

Then by the Area Formula

$$\begin{aligned}\int_\Omega \left(\sum_{k=1}^{n+1}\left[\frac{\partial(f^1,\ldots,f^{k-1},f^{k+1},\ldots,f^{n+1})}{\partial x_1,\ldots,x_n}\right]^2\right)^{\frac{1}{2}} d\mathcal{L}^n(x) &= \int_\Omega Jf(x)\,d\mathcal{L}^n(x)\\ &= \int_{\mathbb{R}^{n+1}} \mathcal{H}^0(\Omega\cap f^{-1}(y))\,d\mathcal{H}^n(y)\\ &= \mathcal{H}^n(S).\end{aligned}$$

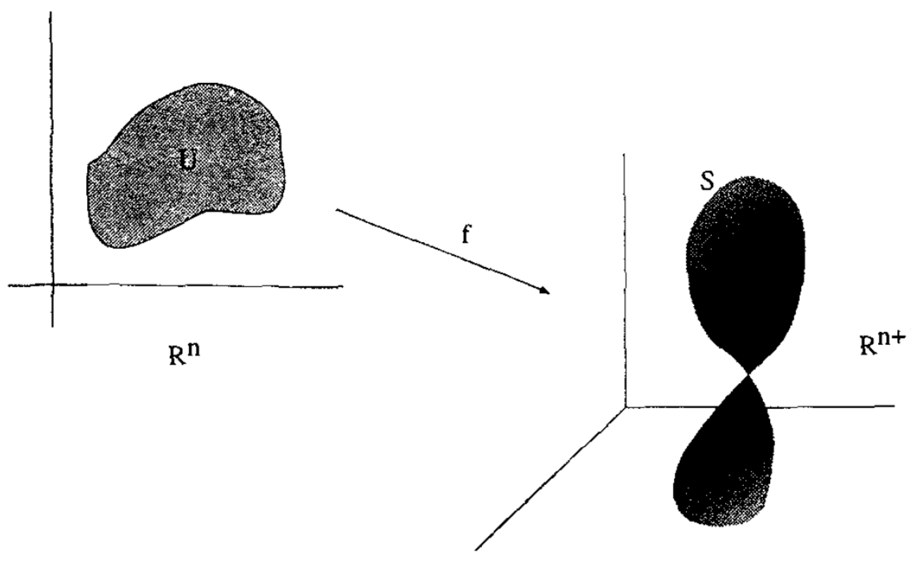

FIGURE 3.3.4. Surface Area of a Parametric Hypersurface.

Example 3.3.4 (Submanifolds). *Let $M\subset\mathbb{R}^m$ be a Lipschitz $n-$dimensional embedded submanifold. Suppose that $\Omega\subset\mathbb{R}^n$ and let $f:\Omega\to M$ be coordinates for M. Let $A\subset f(\Omega)$. Let $A\subset f(\Omega)\subset M$, A Borel, and let $B:=f^{-1}(A)\subset\Omega$. Define the metric $g:M\to\mathbb{R}$ on M by*

$$g_{ij}=g\left(\frac{\partial f}{\partial x_i},\frac{\partial f}{\partial x_j}\right):=\frac{\partial f}{\partial x_i}\cdot\frac{\partial f}{\partial x_j},\quad i,j=1,\ldots,n,$$

and

$$g:=\det((g_{ij})_{n\times n}).$$

Then

$$Df\circ(Df)^*=(g_{ij})_{n\times n},$$

and so

$$Jf=(\det(Df\circ(Df)^*))^{\frac{1}{2}}=g^{\frac{1}{2}}.$$

Thus by the Area Formula,

$$\begin{aligned}\int_B g^{\frac{1}{2}}\,d\mathcal{L}^n(x) &= \int_B Jf(x)\,d\mathcal{L}^n(x)\\ &= \int_{\mathbb{R}^m}\mathcal{H}^0(B\cap f^{-1}(y))\,d\mathcal{H}^n(y)\end{aligned}$$

$$= \mathcal{H}^n(A).$$

Here $\mathcal{H}^n(A)$ represents the "volume" of A in M.

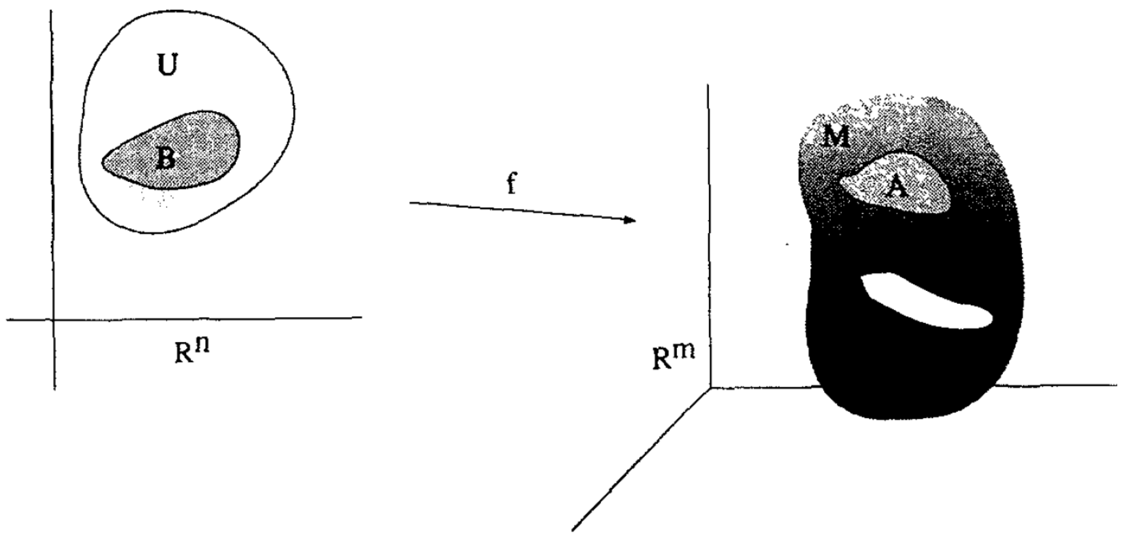

FIGURE 3.3.5. Volume of a Submanifold.

3.4. **The Coarea Formula.** Throughout this section we assume that

$$n \geq m.$$

. . . .

3.4.1. *Preliminaries.*

13.4-1 **Lemma 3.4.1.** *Suppose that $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, $n \geq m$, and $A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$−measurable. Then*

(i) The mapping $y \mapsto \mathcal{H}^{n-m}(A \cap L^{-1}(y))$ is $\mathcal{L}^m$−measurable;
(ii) $\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}(y))\, d\mathcal{L}^m(y) = [\![L]\!]\mathcal{L}^n(A)$.

Proof.
(i). First suppose that $\dim L(\mathbb{R}^n) < m$. In this case $A \cap L^{-1}(y) = \emptyset$ and consequently $\mathcal{H}^{n-m}(A \cap L^{-1}(y)) = 0$ for $\mathcal{L}^m$−a.e. $y \in \mathbb{R}^n$. Also if we write $L = S \circ O^*$ as in the Polar Decomposition Theorem (cf. (3.2.2)) we have $L(\mathbb{R}^n) = S(\mathbb{R}^m)$. Thus $\dim S(\mathbb{R}^m) < m$, and hence $[\![L]\!] = |\det S| = 0$.

(ii). Now suppose that $L = P$, where P is an orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$. In this case, for each $y \in \mathbb{R}^m$, $P^{-1}(y)$ is an $(n-m)$–dimensional affine subspace of $\mathbb{R}^n$, a translation of $P^{-1}(0)$. By Fubini's Theorem,

$$y \mapsto \mathcal{H}^{n-m}(A \cap P^{-1}(y)) \quad \text{is } \mathcal{L}^m - \text{measurable}$$

and
$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap P^{-1}(y))\, d\mathcal{L}^m(y) = \mathcal{L}^n(A), \tag{3.4.1}$$ {eq:3.4-1

as required.

(iii). Now consider the general case that $L : \mathbb{R}^n \to \mathbb{R}^m$, $\dim L(\mathbb{R}^n) = m$. Again applying the Polar Decomposition Theorem (cf. (3.2.2)) we can write
$$L := S \circ O^*,$$
where $S : \mathbb{R}^m \to \mathbb{R}^m$ is symmetric and $O : \mathbb{R}^m \to \mathbb{R}^n$ is orthogonal. Recall that, since S evidently is not singular,
$$[\![L]\!] = |\det S| > 0.$$

(iv). We claim that $O^* = P \circ Q$, where P is the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$ and $Q : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal. To see this, let Q be any orthogonal map of $\mathbb{R}^n$ onto $\mathbb{R}^n$ such that
$$Q^*(x_1, \dots, x_m, 0, \dots, 0) = O(x_1, \dots, x_m)$$
for all $x \in \mathbb{R}^m$. Note that
$$P^*(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0) \in \mathbb{R}^n$$
for all $x \in \mathbb{R}^m$. Thus
$$(Q^* \circ P^*)(x_1, \dots, x_m) = Q * (x_1, \dots, x_m, 0, \dots, 0) = O(x_1, \dots, x_m),$$
so that $O = Q * \circ P^*$, and hence $O^* = P \circ Q$.

(v). Now $L^{-1}(0)$ is an $(n-m)$–dimensional subspace of $\mathbb{R}^n$ and $L^{-1}(y)$ is a translation of $L^{-1}(0)$ for each $y \in \mathbb{R}^m$. Thus by Fubini's Theorem,
$$y \mapsto \mathcal{H}^{n-m}(A \cap L^{-1}(y)) \quad \text{is } \mathcal{L}^m - \text{measurable}$$
and by (3.4.1) we may calculate
$$\begin{aligned}
\mathcal{L}^n(A) &= \mathcal{L}^n(Q(A)) \\
&= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(Q(A) \cap P^{-1}(y))\, d\mathcal{L}^m(y) \\
&= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap Q^{-1} \circ P^{-1}(y))\, d\mathcal{L}^m(y).
\end{aligned}$$
Now set $z := Sy$ to calculate using the change of variables formula (cf. (3.3.2))
$$|\det S|\mathcal{L}^n(A) = \int_A JS(x)\, d\mathcal{L}^n(x) = |\det S|\mathcal{L}^n(A) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap Q^{-1} \circ P^{-1} \circ S^{-1}(z))\, d\mathcal{H}^m(z).$$
but $L = S \circ O^* = S \circ P \circ Q$, and so, since $[\![L]\!] = |\det S|$,
$$[\![L]\!]\mathcal{L}^n(A) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}(z))\, d\mathcal{L}^m(z),$$
as required. The proof is complete. □

l3.4-2 **Lemma 3.4.2.** *Assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz. Let $A \subseteq \mathbb{R}^n$ be $\mathcal{L}^n$–measurable, $n \geq m$. Then*

(i) $f(A)$ is $\mathcal{L}^m$–measurable;
(ii) $A \cap f^{-1}(y)$ is $\mathcal{H}^{n-m}$–measurable for $\mathcal{L}^m$–a.e. $y \in \mathbb{R}^m$;
(iii) The mapping $y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(y))$ is $\mathcal{L}^m$–measurable;

(iv) $\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y) \leq \frac{(\alpha(n-m)\alpha(m))}{\alpha(n)} (\operatorname{Lip} f)^m \mathcal{L}^n(A)$.

Proof.

(i). Assertion (i) is proved exactly in the same way as the corresponding statement of Lemma (3.3.2) (cf. §3.3).

(ii). Next, for each $j = 1, 2, \ldots$, there exist closed balls $\{B_i^j\}_{i=1}^{+\infty}$ such that

$$A \subset \bigcup_{i=1}^{+\infty} B_i^j, \quad \operatorname{diam} B_i^j \leq \frac{1}{j},$$

and

$$\sum_{i=1}^{+\infty} \mathcal{L}^n(B_i^j) \leq \mathcal{L}^n(A) + \frac{1}{j}.$$

Define now $g_i^j : \mathbb{R}^m \to \mathbb{R}$ by

$$g_i^j(x) := \alpha(n-m) \left(\frac{\operatorname{diam} B_i^j}{2} \right)^{n-m} \mathbb{1}_{f(B_i^j)}(x).$$

By assertion (i) of Lemma (3.3.2), g_i^j is $\mathcal{L}^m$−measurable. Note also that for all $y \in \mathbb{R}^m$,

$$\mathcal{H}_{1/j}^{n-m}(A \cap f^{-1}(y)) \leq \sum_{i=1}^{+\infty} g_i^j(y).$$

Indeed, recall that

$$\mathcal{H}_{1/j}^{n-m}(A \cap f^{-1}(y)) = \inf \left\{ \sum_{i=1}^{+\infty} \frac{\alpha(n-m)}{2^{n-m}} (\operatorname{diam} C_i)^{n-m} : A \cap f^{-1}(y) \subseteq \bigcup_{i=1}^{+\infty} C_i, \operatorname{diam} C_i \leq \frac{1}{j} \right\}.$$

On the other hand,

$$g_i^j(y) = \begin{cases} \dfrac{\alpha(n-m)}{2^{n-m}} (\operatorname{diam} B_j^i)^{n-m}, & y \in f^{-1}(B_j^i), \\ 0, & \text{otherwise.} \end{cases}$$

Now since $\operatorname{diam} B_i^j \leq \frac{1}{j}$ and $A \subset \cup_{j=1}^{+\infty} B_i^j$, $\sum_{j=1}^{+\infty} g_i^j(y)$ is contained in the set of series the infimum is taken over. Thus using Fatou's Lemma and the Isodiametric Inequality (cf. Theorem (2.2.1)), we calculate

$$\begin{aligned} &\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^n(y) \\ &\qquad = \int_{\mathbb{R}^m} \lim_{j \to +\infty} \mathcal{H}_{1/j}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y) \\ &\qquad \leq \int_{\mathbb{R}^m} \liminf_{j \to +\infty} \sum_{i=1}^{+\infty} g_i^j(y)\, d\mathcal{L}^m(y) \\ &\qquad \overset{F.L.}{\leq} \liminf_{j \to +\infty} \sum_{i=1}^{+\infty} \int_{\mathbb{R}^m} g_i^j(y)\, d\mathcal{L}^m(y) \end{aligned}$$

$$
\begin{aligned}
&= \liminf_{j\to+\infty} \sum_{i=1}^{+\infty} \alpha(n-m)\left(\frac{\operatorname{diam} B_i^j}{2}\right)^{n-m} \mathcal{L}^m(f(B_i^j)) \\
&\le \liminf_{j\to+\infty} \sum_{i=1}^{+\infty} \alpha(n-m)\left(\frac{\operatorname{diam} B_i^j}{2}\right)^{n-m} \alpha(m)\left(\frac{\operatorname{diam} f(B_i^j)}{2}\right)^{m} \\
&= \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} \liminf_{j\to+\infty} \sum_{i=1}^{+\infty} \left(\frac{\operatorname{diam} f(B_i^j)}{\operatorname{diam} B_i^j}\right)^{m} \alpha(n)\left(\frac{\operatorname{diam} B_i^j}{2}\right)^{n} \\
&\le \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\operatorname{Lip} f)^m \liminf_{j\to+\infty} \sum_{i=1}^{+\infty} \mathcal{L}^n(B_i^j) \\
&\le \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\operatorname{Lip} f)^m \mathcal{L}^n(A).
\end{aligned}
$$

Thus

$$
\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A\cap f^{-1}(y))\, d\mathcal{L}^m(y) \le \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\operatorname{Lip} f)^m \mathcal{L}^n(A). \tag{3.4.2}
$$

{eq:3.4-2

This will prove assertion (iv) once we establish (ii) and (iii).

(iii). *Case #1: A is compact.*

Fix $t \ge 0$, and for each positive integer i, let U_i be the set of all points $y \in \mathbb{R}^m$ for which there exist finitely many open sets $S_1, \dots, S_l$ such that

$$
\begin{cases}
A\cap f^{-1}(y) \subset \bigcup_{j=1}^{l} S_j, \\
\operatorname{diam} S_j \le \dfrac{1}{i}, \quad j = 1, 2, \dots, l, \\
\sum\limits_{j=1}^{l} \alpha(n-m)\left(\dfrac{\operatorname{diam} S_j}{2}\right)^{n-m} \le t + \dfrac{1}{i}.
\end{cases}
$$

(iv). We claim that U_i is open. To see this, assume that $y \in U_i$. Then $A\cap f^{-1}(y) \subset \cup_{j=1}^{l} S_j$, as above. Then since f is continuous and A is compact,

$$
A\cap f^{-1}(z) \subset \bigcup_{j=1}^{l} S_j
$$

for all z sufficiently close to y.

(v). We next claim that

$$
\{y \in \mathbb{R}^m : \mathcal{H}^{n-m}(A\cap f^{-1}(y)) \le t\} = \bigcap_{i=1}^{+\infty} U_i,
$$

and hence the LHS is a Borel set.

Let $\mathcal{H}^{n-m}(A\cap f^{-1}(y)) \le t$. Then for each $\delta > 0$,

$$
\mathcal{H}^{n-m}_{\delta}(A\cap f^{-1}(y)) \le t.
$$

Given i, choose $\delta \in (0, \frac{1}{i})$. Then there exist sets $\{S_j\}_{j=1}^{+\infty}$ such that

$$\begin{cases} A \cap f^{-1}(y) \subset \bigcup_{j=1}^{+\infty} S_j, \\ \operatorname{diam} S_j \leq \delta < \dfrac{1}{i}, \\ \sum\limits_{j=1}^{+\infty} \alpha(n-m) \left(\dfrac{\operatorname{diam} S_j}{2} \right)^{n-m} < t + \dfrac{1}{i}. \end{cases}$$

We may assume that the sets S_j, $j = 1, 2, \ldots$, are open. Since $A \cap f^{-1}(y)$ is compact, a finite subcollection $\{S_1, \ldots, S_l\}$ covers $A \cap f^{-1}(y)$, and hence $y \in U_i$. We may apply the same argument for each $i = 1, 2, \ldots$, and thus

$$\{y \in \mathbb{R}^m : \mathcal{H}^{n-m}(A \cap f^{-1}(y)) \leq t\} \subset \bigcap_{i=1}^{+\infty} U_i.$$

Now let $y \in \cap_{i=1}^{+\infty} U_i$. Then for each i,

$$\begin{aligned} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) &\leq \mathcal{H}^{n-m}_{1/i} \left(\bigcup_{j=1}^{l} S_j \right) \\ &\leq t + \frac{1}{i}, \end{aligned}$$

and so

$$\mathcal{H}^{n-m}(A \cap f^{-1}(y)) \leq t.$$

Therefore

$$\{y \in \mathbb{R}^m : \mathcal{H}^{n-m}(A \cap f^{-1}(y)) \leq t\} = \bigcap_{i=1}^{+\infty} U_i,$$

as required.

(vi). In view of (v), for compact sets A, the mapping

$$y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(y))$$

is Borel measurable, and thus $\mathcal{H}^{n-m}-$measurable.

(vii). *Case #2: A is open.*

If A is open, there exist compact sets $K_1 \subset K_2 \subset \cdots \subset A$ such that

$$A = \bigcup_{i=1}^{+\infty} K_i.$$

This is an increasing sequence, and so for each $y \in \mathbb{R}^m$,

$$\mathcal{H}^{n-m}(A \cap f^{-1}(y)) = \lim_{i \to +\infty} \mathcal{H}^{n-m}(K_i \cap f^{-1}(y)).$$

Thus the mapping

$$y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(y))$$

is Borel measurable, as needed.

(viii). *Case #3: $\mathcal{L}^n(A) < +\infty$.*

In this case there exist open sets $V_1 \supset V_2 \supset \cdots \supset A$ such that

$$\lim_{i \to +\infty} \mathcal{L}^n(V_i \setminus A) = 0, \qquad \mathcal{L}^n(V_1) < +\infty.$$

Now

$$\begin{aligned}\mathcal{H}^{n-m}(V_i \cap f^{-1}(y)) &= \mathcal{H}^{n-m}((A \cup (V_i \setminus A)) \cap f^{-1}(y)) \\ &\leq \mathcal{H}^{n-m}(A \cap f^{-1}(y)) + \mathcal{H}^{n-m}((V_i \setminus A) \cap f^{-1}(y)),\end{aligned}$$

and thus by (3.4.2),

$$\begin{aligned}&\limsup_{i\to+\infty} \int_{\mathbb{R}^m} |\mathcal{H}^{n-m}(V_i \cap f^{-1}(y)) - \mathcal{H}^{n-m}(A \cap f^{-1}(y))| \, d\mathcal{L}^m(y) \\ &\qquad \leq \limsup_{i\to+\infty} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}((V_i \setminus A) \cap f^{-1}(y)) \, d\mathcal{L}^m(y) \\ &\qquad \leq \limsup_{i\to+\infty} \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\operatorname{Lip} f)^m \mathcal{L}^n(V_i \setminus A) \\ &\qquad = 0.\end{aligned}$$

Consequently

$$\mathcal{H}^{n-m}(V_i \cap f^{-1}(y)) \to \mathcal{H}^{n-m}(A \cap f^{-1}(y))$$

for $\mathcal{L}^m-$a.e. $y \in \mathbb{R}^m$, and so according to (vii), the mapping

$$y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(y))$$

is $\mathcal{L}^m-$measurable, being the pointwise a.e. limit of the mappings

$$y \mapsto \mathcal{H}^{n-m}(V_i \cap f^{-1}(y)).$$

In addition, we see that $\mathcal{H}^{n-m}((V_i \setminus A) \cap f^{-1}(y)) \to 0$ for $\mathcal{L}^m-$a.e. $y \in \mathbb{R}^m$, and so $A \cap f^{-1}(y)$ is $\mathcal{H}^{n-m}$ measurable for $\mathcal{L}^m-$a.e. $y \in \mathbb{R}^m$.

(ix). *Case #4:* $\mathcal{L}^n(A) = +\infty$.

In this case we may write A as a union of an increasing sequence of bounded $\mathcal{L}^n-$measurable sets and apply (viii) to prove that

$$A \cap f^{-1}(y) \quad \text{is } \mathcal{H}^{n-m} - \text{measurable for } \mathcal{L}^m - \text{a.e.} y \in \mathbb{R}^m,$$

and

$$y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(y))$$

is $\mathcal{L}^m-$measurable.

(x). Parts (iii) through (ix) prove assertions (ii) and (iii) of the theorem. In view of (3.4.2), this proves assertion (iv) as well. The proof is complete. □

Remark. *A proof similar to that of assertion (iv) of Lemma (3.4.2) shows that*

$$\int_{\mathbb{R}^m} \mathcal{H}^k(A \cap f^{-1}(y)) \, d\mathcal{H}^l(y) \leq \frac{\alpha(k)\alpha(l)}{\alpha(k+l)} (\operatorname{Lip} f)^l \mathcal{H}^{k+l}(A)$$

for each $A \subseteq \mathbb{R}^m$.

Lemma 3.4.3. *Let* $t > 1$, *assume that* $g : \mathbb{R}^n \to \mathbb{R}^n$ *is Lipschitz, and set*

$$B := \{x \in \mathbb{R}^n : Dg(x) \text{ exists}, \, Jg(x) > 0\}.$$

Then there exists a countable collection $\{D_k\}_{k=1}^{+\infty}$ *of Borel subsets of* $\mathbb{R}^n$ *such that*

(i) $\mathcal{L}^n(B \setminus \cup_{k=1}^{+\infty} D_k) = 0$;

(ii) $g|_{D_k}$ *is one–to–one for* $k = 1, 2, \ldots$;

(iii) For each $k = 1, 2, \ldots$, there exists a symmetric automorphism $S_k : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\operatorname{Lip}(S_k^{-1} \circ (g|_{D_k})) \le t, \qquad \operatorname{Lip}((g|_{D_k})^{-1} \circ S_k) \le t,$$
$$t^{-n}|\det S_k| \le Jg|_{D_k} \le t^n|\det S_k|.$$

Proof.

(i). We may apply Lemma (3.3.3) (cf. §3.3) to find Borel sets $\{E_k\}_{k=1}^{+\infty}$ and symmetric automorphisms $T_k : \mathbb{R}^n \to \mathbb{R}^n$ such that

(i) $B = \cup_{k=1}^{+\infty} E_k$,

(ii) $g|_{E_k}$ is one–to–one,

(iii)
$$\begin{cases} \operatorname{Lip}((g|_{E_k}) \circ T_k^{-1}) \le t, \quad \operatorname{Lip}(T_k \circ (g|_{E_k})^{-1}) \le t, \\ t^{-n}|\det T_k| \le Jg|_{E_k} \le t^n|\det T_k|, \quad k = 1, 2, \ldots. \end{cases}$$

By (iii), $(g|_{E_k})^{-1}$ is Lipschitz and thus by Theorem (3.1.1) (cf. §3.1.1, extension of Lipschitz functions) there exists a Lipschitz mapping $g_k : \mathbb{R}^n \to \mathbb{R}^n$ such that $g_k = (h|_{E_k})^{-1}$ on $g(E_k)$.

(ii). We claim that $Jg_k > 0$ $\mathcal{L}^n - a.e.$ on $g(E_k)$. To see this, first note that since $g_k \circ g(x) = x$ for $x \in E_k$, Corollary (3.1.1) (cf. §3.1.2) implies

$$Dg_k(g(x)) \circ Dg(x) = I, \quad \mathcal{L}^n - \text{a.e. on } E_k,$$

and so

$$Jg_k(g(x))Jg(x) = 1 \quad \mathcal{L}^n - \text{a.e. on } E_k.$$

In view of (iii), this implies $Jg_k(g(x)) > 0$ for $\mathcal{L}^n$–a.e. $x \in E_k$, and (ii) follows because g is Lipschitz.

(iii). Now applying Lemma (3.3.3) (cf. §3.3) to g_k, there exist Borel sets $\{F_j^k\}_{j=1}^{+\infty}$ and symmetric automorphisms $\{R_j^k\}_{j=1}^{+\infty}$ such that

(i) $\mathcal{L}^n \left(g(E_k) - \cup_{j=1}^{+\infty} F_j^k \right) = 0$,

(ii) $g_k|_{F_j^k}$ is one–to–one,

(iii)
$$\begin{cases} \operatorname{Lip}\left((g_k|_{F_j^k}) \circ (R_j^k)^{-1} \right) \le t, \quad \operatorname{Lip}\left(R_j^k \circ (g_k|_{F_j^k})^{-1} \right) \le t, \\ t^{-n}\left|\det R_j^k\right| \le Jg_k|_{F_j^k} \le t^n\left|\det R_j^k\right|, \quad k = 1, 2, \ldots. \end{cases}$$

Put

$$D_j^k := E_k \cap g^{-1}(F_j^k), \quad S_j^k := (R_j^k)^{-1}, \quad k = 1, 2, \ldots.$$

(iv). We next claim that $\mathcal{L}^n \left(B \setminus \cup_{k,j=1}^{+\infty} D_j^k \right) = 0$. Note that

$$\begin{aligned} g_k \left(g(E_k) \setminus \bigcup_{j=1}^{+\infty} F_j^k \right) &= g^{-1} \left(g(E_k) \setminus \bigcup_{j=1}^{+\infty} F_j^k \right) \\ &= E_k \setminus \bigcup_{j=1}^{+\infty} D_j^k. \end{aligned}$$

Thus, by (i) and the fact that the image of a set of Lebesgue measure zero has Lebesgue measure zero,

$$\mathcal{L}^n \left(E_k \setminus \bigcup_{j=1}^{+\infty} D_j^k \right) = 0, \quad k = 1, 2, \ldots.$$

By (i) in part (i), this proves (iv).

(v). Clearly (ii) in part (i) implies that $g|_{D_j^k}$ is one–to–one, for $D_j^k \subseteq E_k$, $k = 1, 2, \ldots$.

(vi). We lastly claim that for $k, j = 1, 2, \ldots$, we have

$$\begin{gathered}\operatorname{Lip}((S_j^k)^{-1} \circ (g|_{D_j^k})) \le t, \quad \operatorname{Lip}((g|_{D_j^k})^{-1} \circ S_j^k) \le t, \\ t^{-n} \left|\det S_j^k\right| \le Jg|_{D_j^k} \le t^n \left|\det S_j^k\right|.\end{gathered}$$

Observe that

$$\begin{aligned}\operatorname{Lip}((S_j^k)^{-1} \circ (g|_{D_j^k})) &= \operatorname{Lip}(R_j^k \circ (g|_{D_j^k})) \\ &\le \operatorname{Lip}(R_j^k \circ (g_k|_{F_j^k})^{-1}) \\ &\le t,\end{aligned}$$

because $D_j^k \subseteq g^{-1}(F_j^k)$. Similarly

$$\begin{aligned}\operatorname{Lip}((g|_{D_j^k})^{-1} \circ S_j^k) &= \operatorname{Lip}((g|_{D_j^k})^{-1} \circ (R_j^k)^{-1}) \\ &\le \operatorname{Lip}((g_k|_{F_j^k}) \circ (R_j^k)^{-1}) \\ &\le t.\end{aligned}$$

Moreover, as noted above,

$$Jg_k(g(x))Jg(x) = 1 \quad \mathcal{L}^n-\text{a.e. on} D_j^k.$$

Thus (iii) in part (iii) of the proof implies

$$t^{-n}|\det S_j^k| = t^{-n}|\det R_j^k|^{-1} \le Jg|_{D_j^k} \le t^n|\det R_j^k|^{-1} = t^n|\det S_j^k|,$$

as required. The proof is complete. □

. . . .

3.4.2. *Proof of the Coarea Formula.*

t3.4-1 **Theorem 3.4.1** (Coarea Formula). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \ge m$. Then for each $\mathcal{L}^n$−measurable set $A \subseteq \mathbb{R}^n$,*

$$\int_A Jf(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y).$$

Remark.

(i) The Coarea Formula allows us to integrate $Jf(x)$ over A by integrating the $(n-m)$−dimensional Hausdorff measure of the fibers of f.

(ii) Observe that the Coarea Formula is a kind of "curvilinear" generalization of Fubini's Theorem.

(iii) Applying the Coarea Formula to $A := \{x \in \mathbb{R}^n : Jf(x) = 0\}$, we find

$$\mathcal{H}^{n-m}(\{x \in \mathbb{R}^n : Jf(x) = 0\} \cap f^{-1}(y)) = 0 \tag{3.4.3}$$ {eq:3.4-3

for $\mathcal{L}^m$−a.e. $y \in \mathbb{R}^m$. This is a weak variant of the Morse–Sard Theorem, which asserts

$$\{x \in \mathbb{R}^n : Jf(x) = 0\} \cap f^{-1}(y) = \emptyset$$

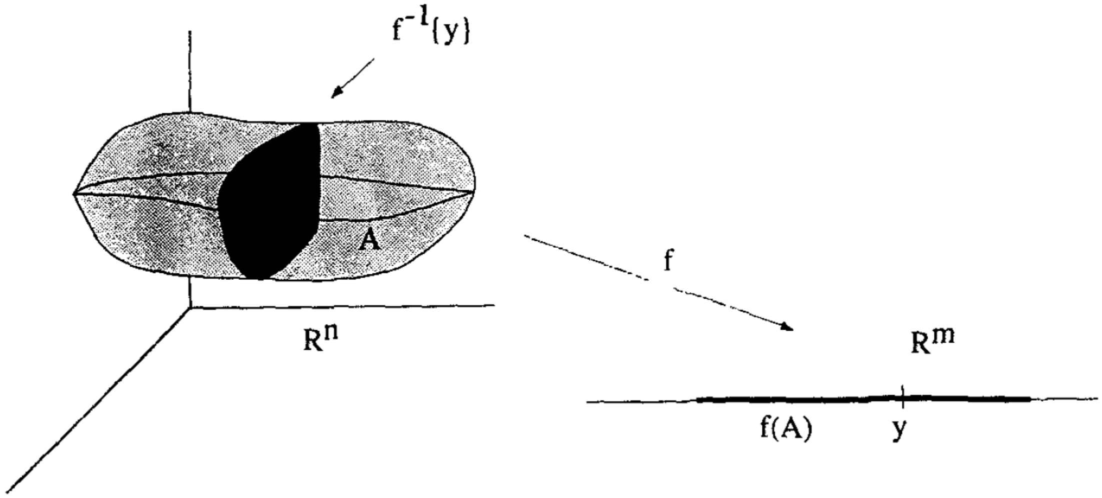

FIGURE 3.4.1. The Coarea Formula.

for $\mathcal{L}^m-$a.e. $y \in \mathbb{R}^m$, provided that $f \in \mathcal{C}^k(\mathbb{R}^n;\mathbb{R}^m)$, for

$$k = 1 + n - m.$$

On the other hand, (3.4.3) required only that f is Lipschitz.

Proof.

(i). By Rademacher's Theorem (cf. Theorem (3.1.2)) and Lemma (3.4.2), we may assume that $Df(x)$, and thus $Jf(x)$, exist for all $x \in A$ and that $\mathcal{L}^n(A) < +\infty$.

(ii). *Case #1:* $A \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$.

For each $\lambda \in \Lambda(n, n-m)$, write

$$f := q \circ h_\lambda,$$

where

$$h_\lambda : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^{n-m}, \quad h_\lambda(x) := (f(x), P_\lambda(x)), \quad x \in \mathbb{R}^n,$$

and

$$q : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m, \quad q(y,z) := y, \quad y \in \mathbb{R}^m, z \in \mathbb{R}^{n-m},$$

and P_λ is the projection

$$P_\lambda(x_1, \ldots, x_n) := (x_{\lambda(1)}, \ldots, x_{\lambda(n-m)})$$

(cf. §3.2.1). Set

$$\begin{aligned} A_\lambda &:= \{x \in A : \det Dh_\lambda \neq 0\} \\ &= \{x \in A : P_\lambda|_{[Df(x)]^{-1}(0)} \text{ is injective}\}. \end{aligned}$$

Now

$$A = \bigcup_{\lambda \in \Lambda(n,n-m)} A_\lambda,$$

and therefore we may as well for simplicity assume that $A = A_\lambda$ for some $\lambda \in \Lambda(n, n-m)$.

(iii). Fix $t > 1$. Applying Lemma (3.4.3) to $h := h_\lambda$, we obtain disjoint Borel sets $\{D_k\}_{k=1}^{+\infty}$ and symmetric automorphisms $\{S_k\}_{k=1}^{+\infty}$ such that

(i) $\mathcal{L}^n(A \setminus \cup_{k=1}^{+\infty} D_k) = 0$;

(ii) $h|_{D_k}$ is one–to–one for $k = 1, 2, \ldots$;

(iii) For each $k = 1, 2, \ldots$,

$$\operatorname{Lip}(S_k^{-1} \circ (h|_{D_k})) \leq t, \qquad \operatorname{Lip}((h|_{D_k})^{-1} \circ S_k) \leq t,$$
$$t^{-n}|\det S_k| \leq Jh_{D_k} \leq t^n|\det S_k|.$$

Set $G_k := A \cap D_k$.

(iv). We claim that

$$t^{-n}[\![q \circ S_k]\!] \leq Jf|_{G_k} \leq t^n[\![q \circ S_k]\!].$$

To see this, first note that since $f = q \circ h$, we have $\mathcal{L}^n$−a.e. that

$$\begin{aligned} Df &= Dq(h) \cdot Dh = q \circ Dh \\ &= q \circ S_k \circ S_k^{-1} \circ Dh \\ &= q \circ S_k \circ D(S_k^{-1} \circ h) \\ &= q \circ S_k \circ C, \end{aligned}$$

where $C := D(S_k^{-1} \circ h)$.

Thus by Lemma (3.4.3),

$$t^{-1} \leq \operatorname{Lip}(S_k^{-1} \circ h) = \operatorname{Lip}(C) \leq t \quad \text{on } G_k. \tag{3.4.4}$$

{eq:3.4-4

Now write

$$\begin{aligned} Df &:= S \circ O^*, \\ q \circ S_k &:= T \circ P^* \end{aligned}$$

for symmetric maps $S, T : \mathbb{R}^m \to \mathbb{R}^n$ and orthogonal maps $O, P : \mathbb{R}^m \to \mathbb{R}^n$ (cf. Theorem (3.2.2)).

We have then

$$S \circ O^* = T \circ P^* \circ C. \tag{3.4.5}$$

{eq:3.4-5

Consequently

$$S = T \circ P^* \circ C \circ O.$$

Since $G_k \subset A \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$, $\det S \neq 0$ and thus $\det T \neq 0$.

Thus if $v \in \mathbb{R}^m$, we have by (3.4.4)

$$\begin{aligned} |T^{-1} \circ Sv| &= |T^{-1} \circ T \circ P^* \circ C \circ Ov| \\ &= |P^* \circ C \circ Ov| \\ &\leq |C \circ Ov| \\ &\leq t|Ov| \\ &= t|v|. \end{aligned}$$

Therefore

$$(T^{-1} \circ S)(B(0,1)) \subset B(0,t),$$

and so

$$Jf = |\det S| \leq t^n|\det T| = t^n[\![q \circ S_k]\!].$$

Similarly, if $v \in \mathbb{R}^m$, we have by (3.4.5) and (3.4.4)

$$\begin{aligned} |S^{-1} \circ Tv| &= |O^* \circ C^{-1} \circ P \circ T^{-1} \circ Tv| \\ &= |O^* \circ C^{-1} \circ Pv| \end{aligned}$$

$$
\begin{aligned}
&\leq |C^{-1} \circ Pv| \\
&\leq t|Pv| \\
&= t|v|.
\end{aligned}
$$

Thus

$$(S^{-1} \circ T)(B(0,1)) \subset B(0,t),$$

so evidently

$$[\![q \circ S_k]\!] = |\det T| \leq t^n |\det S| = t^n Jf.$$

This establishes the claim.

(v). We now calculate by Lemmas (3.4.1) and (3.4.3) and Theorem (2.4.1)

$$
\begin{aligned}
t^{-3n+m} &\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(G_k \cap f^{-1}(y))\, d\mathcal{L}^m(y) \\
&= t^{-3n+m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(h^{-1}(h(G_k) \cap q^{-1}(y)))\, d\mathcal{L}^m(y) \\
&\leq t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(S_k^{-1}(h(G_k) \cap q^{-1}(y)))\, d\mathcal{L}^m(y) \\
&= t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(S_k^{-1} \circ h(G_k) \cap (q \circ S_k)^{-1}(y))\, d\mathcal{L}^m(y) \\
&= t^{-2n} [\![q \circ S_k]\!] \mathcal{L}^n(S_k^{-1} \circ h(G_k)) \\
&\leq t^{-n} [\![q \circ S_k]\!] \mathcal{L}^n(G_k) \\
&= \int_{G_k} t^{-n} [\![q \circ S_k]\!]\, d\mathcal{L}^n(x) \\
&\leq \int_{G_k} Jf(x)\, d\mathcal{L}^n(x) \\
&\leq \int_{G_k} t^n [\![q \circ S_k]\!]\, d\mathcal{L}^n(x) \\
&= t^n [\![q \circ S_k]\!] \mathcal{L}^n(G_k) \\
&\leq t^{2n} [\![q \circ S_k]\!] \mathcal{L}^n(S_k^{-1} \circ h(G_k)) \\
&= t^{2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(S_k^{-1} \circ h(G_k) \cap (q \circ S_k)^{-1})\, d\mathcal{L}^m(y) \\
&\leq t^{3n-m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(h^{-1}(h(G_k) \cap q^{-1}(y)))\, d\mathcal{L}^m(y) \\
&= t^{3n-m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(G_k \cap f^{-1}(y))\, d\mathcal{L}^m(y).
\end{aligned}
$$

Since

$$\mathcal{L}^n\left(A \setminus \bigcup_{k=1}^{+\infty} G_k\right) = 0,$$

we may sum on k to obtain

$$t^{-3n+m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y) \leq \int_A Jf(x)\, d\mathcal{L}^n(x)$$

$$\leq t^{3n-m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y).$$

Letting $t \to 1^+$, we conclude that

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y) = \int_A Jf(x)\, d\mathcal{L}^n(x),$$

which completes the proof for this case.

(vi). *Case #2:* $A \subset \{x \in \mathbb{R}^n : Jf(x) = 0\}$.

In this case fix $\epsilon > 0$ and define

$$g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m, \qquad g(x,y) := f(x) + \epsilon y,$$
$$p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m, \qquad p(x,y) := y, \qquad x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

We claim that there exists a constant $C > 0$ such that

$$0 < Jg(x) \leq C\epsilon$$

for all $x \in A$. Notice that

$$Dg(x) = (Df(x), \epsilon I).$$

By the Binet–Cauchy Formula (cf. (3.2.4)), $Jg(x)^2$ equals the sum of squares of all $(m \times m)$ subdeterminants of $Dg(x)$, so

$$Jg(x)^2 \geq \epsilon^{2m} > 0.$$

Moreover, since $|Df| \leq \mathrm{Lip}(f) < +\infty$, the Binet–Cauchy formula also gives

$$Jg(x) = Jf(x)^2 + \{\text{sum of squares of terms involving at least one } \epsilon\} \leq C\epsilon^2$$

for each $x \in A$. Thus

$$\epsilon^m \leq Jg = [\![Dg]\!] \leq C\epsilon.$$

(vii). Observe that

$$\begin{aligned}
&\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y) \\
&\quad = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y - \epsilon w))\, d\mathcal{L}^m(y) \quad \text{for all } w \in \mathbb{R}^m \\
&\quad = \frac{1}{\alpha(m)} \int_{B(0,1)} \int_{\mathbb{R}^m} \mathcal{H}^{n-1}(A \cap f^{-1}(y - \epsilon w))\, d\mathcal{L}^m(y)\, d\mathcal{L}^m(w).
\end{aligned}$$

(viii). Fix $y, w \in \mathbb{R}^m$, and set $B := A \times B(0,1) \subset \mathbb{R}^{n+m}$. We claim that

$$B \cap g^{-1}(y) \cap p^{-1}(w) = \begin{cases} \emptyset, & w \notin B(0,1), \\ (A \cap f^{-1}(y - \epsilon w)) \times \{w\}, & w \in B(0,1). \end{cases}$$

To see this, note that we have $(x,z) \in B \cap g^{-1}(y) \cap p^{-1}(w)$ if and only if

$$x \in A, \quad z \in B(0,1), \quad f(x) + \epsilon z = y, \quad z = w.$$

Moreover, this holds if and only if

$$x \in A, \quad z = w \in B(0,1), \quad f(x) = y - \epsilon w.$$

Finally, the above holds if and only if

$$w \in B(0,1), \quad (x,z) \in (A \cap f^{-1}(y - \epsilon w)) \times \{w\}.$$

This proves (viii).

(ix). We use (viii) to continue the calculation from (vii), and obtain by Lemma (3.4.2) l3.4-2 and Case #1

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y) \\
&\qquad = \frac{1}{\alpha(m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(B \cap g^{-1}(y) \cap p^{-1}(w))\, d\mathcal{L}^m(w)\, d\mathcal{L}^m(y) \\
&\qquad \le \frac{1}{\alpha(m)} \frac{\alpha(m)\alpha(n-m)}{\alpha(n)} (\operatorname{Lip} p)^m \int_{\mathbb{R}^m} \mathcal{H}^n(B \cap g^{-1}(y))\, d\mathcal{L}^m(y) \\
&\qquad = \frac{\alpha(n-m)}{\alpha(n)} \int_{\mathbb{R}^m} \mathcal{H}^n(B \cap g^{-1}(y))\, d\mathcal{L}^m(y) \\
&\qquad = \frac{\alpha(n-m)}{\alpha(n)} \int_B Jg(x,z)\, d\mathcal{L}^n(x)\, d\mathcal{L}^m(z) \\
&\qquad \le \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} \mathcal{L}^n(A) \sup_B Jg(x,z) \\
&\qquad \le C\mathcal{L}^n(A)\epsilon.
\end{aligned}
$$

Letting $\epsilon \to 0$, we obtain

$$
\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y) = 0 = \int_A Jf(x)\, d\mathcal{L}^n(x),
$$

as required.

(x). In the general case we write $A := A_1 \cup A_2$, where $A_1 \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$ and $A_2 \subset \{x \in \mathbb{R}^n : Jf(x) = 0\}$, and apply Cases #1 and #2 above. The proof is complete. $\square$

. . . .

3.4.3. *Change of Variables Formula.*

t3.4-2 **Theorem 3.4.2.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \ge m$. Then for each $\mathcal{L}^n$−integrable function $g : \mathbb{R}^n \to \mathbb{R}$,*

$$
g|_{f^{-1}(y)} \quad \text{is } \mathcal{H}^{n-m} - \text{integrable for } \mathcal{L}^m - \text{a.e. } y \in \mathbb{R}^m,
$$

and

$$
\int_{\mathbb{R}^n} g(x) Jf(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \left[\int_{f^{-1}(y)} g\, d\mathcal{H}^{n-m} \right] d\mathcal{L}^m(y).
$$

Proof.

(i). *Case #1:* $g \ge 0$.

Define the sequence $\{s_j\}_{j=1}^{+\infty}$ by

$$
s_j(x) := \sum_{k=0}^{j2^j} \frac{k}{2^j} \mathbb{1}_{g^{-1}[\frac{k}{2^j}, \frac{k+1}{2^j})}(x) + j \mathbb{1}_{g^{-1}[j,+\infty]}(x).
$$

Recall that $s_j \to g$ as $j \to +\infty$ and

$$
0 \le s_1 \le s_2 \le \cdots .
$$

Hence, by the Monotone Convergence Theorem,

$$
\begin{aligned}
\int_{\mathbb{R}^n} g(x)Jf(x)\,d\mathcal{L}^n(x) &= \int_{\mathbb{R}^n} \lim_{j\to+\infty} s_j(x)Jf(x)\,d\mathcal{L}^n(x)\\
&\overset{MCT}{=} \lim_{j\to+\infty}\int_{\mathbb{R}^n} s_n(x)Jf(x)\,d\mathcal{L}^n(x)\\
&= \lim_{j\to+\infty}\int_{\mathbb{R}^n}\left(\sum_{k=0}^{j2^j}\frac{k}{2^j}\mathbb{1}_{g^{-1}[\frac{k}{2^j},\frac{k+1}{2^j})}(x) + j\mathbb{1}_{g^{-1}[j,+\infty]}(x)\right)Jf(x)\,d\mathcal{L}^n(x)\\
&\overset{B.L.}{=} \lim_{j\to+\infty}\sum_{k=0}^{j2^j}\frac{k}{2^j}\int_{g^{-1}[\frac{k}{2^j},\frac{k+1}{2^j})} Jf(x)\,d\mathcal{L}^n(x)\\
&= \lim_{j\to+\infty}\sum_{k=0}^{j2^j}\frac{k}{2^j}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}\left(g^{-1}\left[\frac{k}{2^j},\frac{k+1}{2^j}\right)\cap f^{-1}(y)\right)\,d\mathcal{L}^m(y)\\
&\overset{B.L.}{=} \lim_{j\to+\infty}\int_{\mathbb{R}^m}\sum_{k=0}^{j2^j}\frac{k}{2^j}\mathcal{H}^{n-m}\left(g^{-1}\left[\frac{k}{2^j},\frac{k+1}{2^j}\right)\cap f^{-1}(y)\right)\,d\mathcal{L}^m(y)\\
&\overset{MCT}{=} \int_{\mathbb{R}^m}\lim_{j\to+\infty}\sum_{k=0}^{j2^j}\frac{k}{2^j}\mathcal{H}^{n-m}\left(g^{-1}\left[\frac{k}{2^j},\frac{k+1}{2^j}\right)\cap f^{-1}(y)\right)\,d\mathcal{L}^m(y)\\
&= \int_{\mathbb{R}^m}\left[\int_{f^{-1}(y)}\lim_{j\to+\infty}\sum_{k=0}^{j2^j}\frac{k}{2^j}\mathbb{1}_{g^{-1}[\frac{k}{2^j},\frac{k+1}{2^j})}(x)\,d\mathcal{H}^{n-m}(x)\right]\,d\mathcal{L}^m(y)\\
&= \int_{\mathbb{R}^m}\left[\int_{f^{-1}(y)} g(x)\,d\mathcal{H}^{n-m}(x)\right]\,d\mathcal{L}^m(y),
\end{aligned}
$$

as required.

(ii). *Case #2: g is any $\mathcal{L}^n-$integrable function.* In this case, write $g := g^+ - g^-$ and apply Case #1. The proof is complete. $\square$

. . . .

3.4.4. *Applications.*

p3.4-1 **Proposition 3.4.1** (Polar Coordinates). *Let $g:\mathbb{R}^n\to\mathbb{R}$ be $\mathcal{L}^n-$integrable. Then*

$$\int_{\mathbb{R}^n} g(x)\,d\mathcal{L}^n(x) = \int_0^{+\infty}\left[\int_{\partial B(0,r)} g(x)\,d\mathcal{H}^{n-1}(x)\right]\,dr.$$

In particular, we see that

$$\frac{d}{dr}\left[\int_{B(0,r)} g(x)\,d\mathcal{L}^n(x)\right] = \int_{\partial B(0,r)} g(x)\,d\mathcal{H}^{n-1}(x)$$

for $\mathcal{L}^1-$a.e. $r>0$.

Proof. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) := |x|$. Then for all $x \neq 0$, we have

$$Df(x) = \frac{x}{|x|}, \qquad Jf(x) = 1.$$

Thus the Change of Variables Formula (cf. (3.4.2)) gives

$$\begin{aligned} \int_{\mathbb{R}^n} g(x)\, d\mathcal{L}^n(x) &= \int_{\mathbb{R}} \left[\int_{f^{-1}(r)} g(x)\, d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(r) \\ &= \int_0^{+\infty} \left[\int_{\partial B(0,r)} g(x)\, d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(r), \end{aligned}$$

as required.

For the second assertion, observe first that

$$\int_{B(0,r)} g(x)\, d\mathcal{L}^n(x) = \int_0^r \left[\int_{\partial B(0,s)} g(x)\, d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s).$$

Hence, by the Fundamental Theorem of Calculus for Lebesgue Integrals,

$$\frac{d}{dr} \left(\int_{B(0,r)} g(x)\, d\mathcal{L}^n(x) \right) = \int_{\partial B(0,r)} g(x)\, d\mathcal{H}^{n-1}(x).$$

The proof is complete. □

p3.4-2 **Proposition 3.4.2** (Level Sets)**.** *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz. Then*

$$\int_{\mathbb{R}^n} |Df(x)|\, d\mathcal{L}^n(x) = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\{f = t\})\, d\mathcal{L}^1(t).$$

Proof. Noting that $Jf(x) = |Df(x)|$, we have directly by the Coarea Formula

$$\begin{aligned} \int_{\mathbb{R}^n} |Df(x)|\, d\mathcal{L}^n(x) &= \int_{\mathbb{R}} \mathcal{H}^{n-1}(f^{-1}(t))\, d\mathcal{L}^1(t) \\ &= \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\{f = t\})\, d\mathcal{L}^1(t). \end{aligned}$$

The proof is complete. □

Remark. *Compare Proposition (3.4.2) with the Coarea Formula for BV functions which will be proved in §5.5.*

p3.4-3 **Proposition 3.4.3** (Level Sets)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz, with*

$$\operatorname{essinf}_{x \in \mathbb{R}^n} |Df(x)| > 0.$$

Suppose also that $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{L}^n-$integrable. Then

$$\int_{\{f>t\}} g(x)\, d\mathcal{L}^n(x) = \int_t^{+\infty} \left[\int_{\{f=s\}} \frac{g(x)}{|Df(x)|}\, d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s).$$

In particular, we see that

$$\frac{d}{dt} \left[\int_{\{f>t\}} g(x)\, d\mathcal{L}^n(x) \right] = -\int_{\{f=t\}} \frac{g(x)}{|Df(x)|}\, d\mathcal{H}^{n-1}(x).$$

Proof. Again recall that $Jf(x) = |Df(x)|$. Write $E_t := \{x \in \mathbb{R}^n : f(x) > t\}$. By the Change of Variables Formula (cf. (3.4.2)), we have

$$\begin{aligned}
\int_{\{f>t\}} g(x)\, d\mathcal{L}^n(x) &= \int_{\mathbb{R}^n} \frac{g(x)}{|Df(x)|} \mathbb{1}_{E_t}(x) Jf(x)\, d\mathcal{L}^n(x) \\
&= \int_{\mathbb{R}} \left[\int_{f^{-1}(s)} \frac{g(x)}{|Df(x)|} \mathbb{1}_{E_t}(x)\, d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s) \\
&= \int_{-\infty}^{+\infty} \left[\int_{\{f=s\}} \frac{g(x)}{|Df(x)|} \mathbb{1}_{E_t}(x)\, d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s) \\
&= \int_{t}^{+\infty} \left[\int_{\{f=s\}} \frac{g(x)}{|Df(x)|}\, d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s),
\end{aligned}$$

as required.

Applying the Fundamental Theorem for Lebesgue Integrals gives

$$\frac{d}{dt} \left[\int_{\{f>t\}} g(x)\, d\mathcal{L}^n(x) \right] = - \int_{\{f>t\}} \frac{g(x)}{|Df(x)|}\, d\mathcal{H}^{n-1}(x).$$

The proof is complete. □

4. BV Functions and Sets of Finite Perimeter

Throughout this chapter, Ω will denote an open subset of $\mathbb{R}^n$.

4.1. **Definitions; Structure Theorem.**

Definition (Bounded Variation)**.** *A function $f \in L^1(\Omega)$ is said to have* **bounded variation** *in Ω if*

$$\sup\left\{\int_\Omega f \operatorname{div}\phi \, d\mathcal{L}^n(x) : \phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n), |\phi| \leq 1\right\} < +\infty.$$

We will write

$$BV(\Omega)$$

to denote the space of functions of bounded variation on Ω.

Definition ($\|\cdot\|_{BV(\Omega)}$)**.** *If $f \in BV(\Omega)$, we define the norm*

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + \sup\left\{\int_\Omega f \operatorname{div}\phi \, d\mathcal{L}^n : \phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n), |\phi| \leq 1\right\}.$$

Definition (Finite Perimeter)**.** *An $\mathcal{L}^n-$measurable subset $E \subset \mathbb{R}^n$ is said to have* **finite perimeter** *in Ω if*

$$\mathbb{1}_E \in BV(\Omega).$$

We also introduce the local versions of the above concepts.

Definition (Locally Bounded Variation)**.** *A function $f \in \mathcal{L}^1_{\text{loc}}(\Omega)$ is said to have* **locally bounded variation** *in Ω if for each open set $U \subset\subset \Omega$,*

$$\sup\left\{\int_U f \operatorname{div}\phi \, d\mathcal{L}^n(x) : \phi \in \mathcal{C}_c^1(U;\mathbb{R}^n), |\phi| \leq 1\right\} < +\infty.$$

We will write

$$BV_{\text{loc}}(\Omega)$$

to denote the space of functions of locally bounded variation on Ω.

Definition (Locally Finite Perimeter)**.** *An $\mathcal{L}^n-$ measurable subset $E \subset \mathbb{R}^n$ is said to have* **locally finite perimeter** *in Ω if*

$$\mathbb{1}_E \in BV_{\text{loc}}(\Omega).$$

We now present the BV Structure Theorem, which asserts that the weak first partial derivatives of a function $f \in BV(\Omega)$ are Radon measures.

t5.1-1 **Theorem 4.1.1** (Structure Theorem for BV_{loc} Functions)**.** *Let $f \in BV_{\text{loc}}(\Omega)$. Then there exists a Radon measure μ on Ω and a $\mu-$measurable function $\sigma : \Omega \to \mathbb{R}^n$ such that*

(i) $|\sigma(x)| = 1$ *for $\mu-$a.e.* $x \in \Omega$;

(ii) $\int_\Omega f \operatorname{div}\phi \, d\mathcal{L}^n(x) = -\int_\Omega \phi \cdot \sigma \, d\mu$

for all $\phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n)$.

Proof. Define the linear functional

$$L : \mathcal{C}_c^1(\Omega;\mathbb{R}^n) \to \mathbb{R}$$

by

$$L(\phi) := -\int_\Omega f \operatorname{div} \phi \, d\mathcal{L}^n(x)$$

for $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$. Since $f \in BV_{\text{loc}}(\Omega)$, we have for each open set $U \subset\subset \Omega$

$$\sup \left\{ L(\phi) : \phi \in \mathcal{C}_c^1(U; \mathbb{R}^n), |\phi| \le 1 \right\} := C(U) < +\infty.$$

Thus

$$|L(\phi)| \le C(U) \|\phi\|_{L^\infty(U)} \tag{4.1.1}$$

for $\phi \in \mathcal{C}_c^1(U; \mathbb{R}^n)$.

Fix any compact set $K \subset \Omega$, and then choose an open set U such that $K \subset U \subset\subset \Omega$. For each $\phi \in \mathcal{C}_c(\Omega; \mathbb{R}^n)$ with $\operatorname{supp} \phi \subset K$, choose $\phi_k \in \mathcal{C}_c^1(U; \mathbb{R}^n)$, $k = 1, 2, \dots$, so that $\phi_k \to \phi$ uniformly on U. Define

$$\overline{L}(\phi) := \lim_{k \to +\infty} L(\phi_k).$$

By (4.1.1), L is bounded, and thus the above limit exists and is independent of the choice of sequence $\{\phi_k\}_{k=1}^{+\infty}$ converging to ϕ. Since $\mathcal{C}_c^1(U; \mathbb{R}^n)$ is dense in

$$\{\phi \in \mathcal{C}_c(\Omega; \mathbb{R}^n) : \operatorname{supp} \phi \subset K\},$$

we have by the BLT Theorem that L uniquely extends to a bounded linear functional

$$\overline{L} : \mathcal{C}_c(\Omega; \mathbb{R}^n) \to \mathbb{R}$$

and

$$\sup \left\{ \overline{L}(\phi) : \phi \in \mathcal{C}_c(\Omega; \mathbb{R}^n) : |\phi| \le 1, \operatorname{supp} \phi \subset K \right\} < +\infty$$

for each compact set $K \subset \Omega$. By the Riesz Representation Theorem, there exists a Radon measure μ on Ω and a $\mu-$measurable function $\sigma : \Omega \to \mathbb{R}^n$ such that

(i) $|\sigma(x)| = 1$ for $\mu-$a.e. $x \in \Omega$;
(ii) $\overline{L}(\phi) = \int_\Omega \phi \ \cdot \sigma \, d\mu$

for all $\phi \in \mathcal{C}_c(\Omega; \mathbb{R}^n)$. Since $\overline{L}$ is an extension of L from $\mathcal{C}_c^1(\Omega; \mathbb{R}^n)$ to $\mathcal{C}_c(\Omega; \mathbb{R}^n)$, it follows that $\overline{L}(\phi) = L(\phi)$ whenever $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$. Hence,

$$\int_\Omega f \operatorname{div} \phi \, d\mathcal{L}^n(x) = -L(\phi) = -\int_\Omega \phi \cdot \sigma \, d\mu$$

for all $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$. The proof is complete. □

Remark (Notation)**.**

(i) If $f \in BV_{\text{loc}}(\Omega)$, we will write

$$\|Df\|$$

for the measure μ, and

$$[Df] := \|Df\| \llcorner \sigma,$$

where $[Df] = \|Df\| \llcorner \sigma$ denotes that $[Df]$ is the measure with density σ with respect to $\|Df\|$, that is,

$$[Df](K) = \int_K \sigma d \, \|Df\|$$

for all compact sets $K \subset \Omega$. Thus assertion (ii) in the Structure Theorem (4.1.1) reads

$$\int_\Omega f \operatorname{div} \phi \, d\mathcal{L}^n(x) = -\int_\Omega \phi \cdot \sigma \, d\|Df\| = -\int_\Omega \phi \cdot d[Df]$$

for all $\phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n)$.

(ii) Similarly if $f = \mathbb{1}_E$, *and* E *is a set of locally finite perimeter in* Ω, *we write*

$$\|\partial E\|$$

for the measure μ, *and*

$$\nu_E := -\sigma.$$

Consequently the Structure Theorem gives

$$\int_E \operatorname{div}\phi \, d\mathcal{L}^n(x) = \int_\Omega \phi \cdot \nu_E \, d\|\partial E\|$$

for all $\phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n)$.

Remark (More Notation). *If* $f \in BV_{\mathrm{loc}}(\Omega)$, *we write*

$$\mu^i := \|Df\| \llcorner \sigma^i, \qquad i = 1,\dots,n$$

for $\sigma = (\sigma^i,\dots,\sigma^n)$. *By Lebesgue's Decomposition Theorem, we may further set*

$$\mu^i = \mu^i_{ac} + \mu^i_s,$$

where

$$\mu^i_{ac} << \mathcal{L}^n, \qquad \mu^i_s \perp \mathcal{L}^n.$$

Then by the Radon–Nikodym Theorem,

$$\mu^i_{ac} = \mathcal{L}^n \llcorner f^i$$

for some function $f^i \in L^1_{loc}(\Omega)$, $i = 1,\dots,n$. *Write*

$$\begin{cases} \dfrac{\partial f}{\partial x_i} := f^i, \quad i = 1,\dots,n, \\ Df := \left(\dfrac{\partial f}{\partial x_1},\dots,\dfrac{\partial f}{\partial x_n}\right), \\ [Df]_{ac} := (\mu^1_{ac},\dots,\mu^n_{ac}) = \mathcal{L}^n \llcorner Df, \\ [Df]_s := (\mu^1_s,\dots,\mu^n_s). \end{cases}$$

Thus

$$[Df] = [Df]_{ac} + [Df]_s = \mathcal{L}^n \llcorner Df + [Df]_s,$$

so that $Df \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ *is the density of the absolutely continuous part of* $[Df]$.

Remark.

(i) $\|Df\|$ *is the* **variation measure** *of* f, $\|\partial E\|$ *is the* **perimeter measure** *of* E, *and* $\|\partial E\|(\Omega)$ *is the* **perimeter** *of* E *in* Ω.

(ii) If $f \in BV_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$, *then* $f \in BV(\Omega)$ *if and only if* $\|Df\|(\Omega) < +\infty$. *To see this, first let* $f \in BV(\Omega)$. *Then by the Structure Theorem*

$$\begin{aligned} \|Df\|(\Omega) &= \int_\Omega \sigma \, d\|Df\| \\ &= \sup\left\{ \int_\Omega \phi \cdot \sigma \, d\|Df\| : \phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n), |\phi| \le 1 \right\} \\ &= \sup\left\{ -\int_\Omega f \ \operatorname{div}\phi \, d\mathcal{L}^n : \phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n), |\phi| \le 1 \right\} < +\infty \end{aligned}$$

Now if $\|Df\|(\Omega) < +\infty$, we have again by the Structure Theorem

$$\begin{aligned} -\int_\Omega f \operatorname{div}\phi \, d\mathcal{L}^n &= \int_\Omega \phi \cdot \sigma \, d\|Df\| \\ &\leq \int_\Omega d\|Df\| < +\infty, \end{aligned}$$

for all $\phi \in \mathcal{C}^1_c(\Omega;\mathbb{R}^n)$ with $|\phi| \leq 1$, so that $f \in BV(\Omega)$.

In this case we define the BV norm of f by

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + \|Df\|(\Omega).$$

(iii). From the proof of the Riesz Representation Theorem, we see that

$$\begin{aligned} \|Df\|(U) &= \sup\left\{\int_U f \operatorname{div}\phi \, d\mathcal{L}^n : \phi \in \mathcal{C}^1_c(U;\mathbb{R}^n), |\phi| \leq 1\right\}, \\ \|\partial E\|(U) &= \sup\left\{\int_E \operatorname{div}\phi \, d\mathcal{L}^n : \phi \in \mathcal{C}^1_c(U;\mathbb{R}^n), |\phi| \leq 1\right\} \end{aligned}$$

for each $U \subset\subset \Omega$. Here we have used the fact that the dual of the space of vector–valued Radon measures on Ω is $\mathcal{C}_c(\Omega;\mathbb{R}^n)$ along the Hahn–Banach Theorem and Structure Theorem.

Example 4.1.1. *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Then for each $U \subset\subset \Omega$ and $\phi \in \mathcal{C}^1_c(U;\mathbb{R}^n)$, with $|\phi| \leq 1$, we have*

$$\int_\Omega f \operatorname{div}\phi \, d\mathcal{L}^n = -\int_\Omega Df \cdot \phi \, d\mathcal{L}^n \leq \int_U |Df| \, d\mathcal{L}^n < +\infty.$$

Furthermore, if we put

$$\|Df\| = \mathcal{L}^n \llcorner |Df|,$$

and

$$\sigma(x) = \begin{cases} \dfrac{Df(x)}{|Df(x)|}, & Df(x) \neq 0, \\ 0, & Df(x) = 0 \end{cases} \qquad \mathcal{L}^n - a.e.,$$

we see that $\|Df\|$ is a Radon measure and $|\sigma(x)| = 1$ $\mathcal{L}^n$–a.e. Moreover

$$\begin{aligned} -\int_\Omega \phi \cdot \sigma \, d\|Df\| &= -\int_U \phi \cdot \frac{Df}{|Df|}|Df| \, d\mathcal{L}^n \\ &= \int_\Omega f \operatorname{div}\phi \, d\mathcal{L}^n \\ &= \int_U f \operatorname{div}\phi \, d\mathcal{L}^n. \end{aligned}$$

Hence

$$W^{1,1}_{\mathrm{loc}}(\Omega) \subset BV_{\mathrm{loc}}(\Omega),$$

and similarly

$$W^{1,1}(\Omega) \subset BV(\Omega).$$

In particular,

$$W^{1,p}_{\mathrm{loc}}(\Omega) \subset BV_{\mathrm{loc}}(\Omega)$$

for $1 \leq p \leq +\infty$, for $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ implies $W^{1,1}_{\mathrm{loc}}(\Omega)$. That is, each Sobolev function has locally bounded variation.

Example 4.1.2. *Let $\Omega = \mathbb{R}^n$, and let $B = B(0,1)$ be the open unit ball in $\mathbb{R}^n$. Then the BV Structure Theorem gives for each $\phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n)$, $|\phi| \leq 1$,*

$$\int_B \operatorname{div}\phi \, d\mathcal{L}^n = \int_\Omega \mathbb{1}_B \operatorname{div}\phi \, d\mathcal{L}^n = -\int_\Omega \phi \cdot \nu_B \, d\|\partial B\|.$$

On the other hand, by the Divergence Theorem, we obtain

$$\int_B \operatorname{div}\phi \, d\mathcal{L}^n = \int_{\partial B} \phi \cdot \nu \, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(\partial B) < +\infty,$$

where ν denotes the outward–pointing unit normal vector on ∂B. Hence B has finite perimeter in $\mathbb{R}^n$. Moreover we see that if we put

$$\nu_B := \nu,$$

then evidently

$$\|\partial B\| = \mathcal{H}^{n-1} \llcorner \mathbb{1}_{\partial B}.$$

Example 4.1.3. *Let E be a smooth, open subset of $\mathbb{R}^n$ and assume that $\mathcal{H}^{n-1}(\partial E \cap K) < +\infty$ for each $K \subset \Omega$. Then for each $U \subset\subset \Omega$ and each $\phi \in \mathcal{C}_c^1(U;\mathbb{R}^n)$ with $|\phi| \leq 1$, we have by the Divergence Theorem*

$$\int_E \operatorname{div}\phi \, d\mathcal{L}^n(x) = \int_{\partial E} \phi \cdot \nu \, d\mathcal{H}^{n-1}$$

where ν denotes the outward–pointing unit normal along ∂E.

Thus

$$\int_E \operatorname{div}\phi \, d\mathcal{L}^n = \int_{\partial E \cap U} \phi \cdot \nu \, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(\partial E \cap U) < +\infty.$$

That is, E has locally finite perimeter in Ω. Furthermore

$$\|\partial E\|(\Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)$$

and

$$\nu_E = \nu \quad \mathcal{H}^{n-1} - \textit{a.e. on } \partial E \cap \Omega.$$

Thus $\|\partial E\|(\Omega)$ measures the "size" of ∂E in Ω. Since $\mathbb{1}_E \notin W^{1,1}_{\mathrm{loc}}(\Omega)$, we see that

$$W^{1,1}_{\mathrm{loc}}(\Omega) \subsetneq BV_{\mathrm{loc}}(\Omega)$$

and similarly

$$W^{1,1}(\Omega) \subsetneq BV(\Omega).$$

That is, not every function of (locally) bounded variation is a Sobolev function.

Remark. *If $f \in BV_{\mathrm{loc}}(\Omega)$, we can write as above*

$$[Df] = [Df]_{ac} + [Df]_s = \mathcal{L}^n \llcorner Df + [Df]_s.$$

Consequently, $f \in BV_{\mathrm{loc}}(\Omega)$ belongs to $W^{1,p}_{\mathrm{loc}}(\Omega)$ if and only if

$$f \in L^p_{\mathrm{loc}}(\Omega), \quad [Df]_s = 0, \quad Df \in L^p_{\mathrm{loc}}(\Omega).$$

We see by the above remark that the theory of BV functions is more subtle than the theory of Sobolev functions, since we have to keep track of the singular part $[Df]_s$ of the vector–valued measure Df.

4.2. **Approximation and Compactness.**

4.2.1. *Lower Semicontinuity.*

t5.2-1 **Theorem 4.2.1** (Lower Semicontinuity of Variation Measure). *Suppose that* $\{f_k\}_{k=1}^{+\infty} \subset L^1_{\mathrm{loc}}(\Omega)$ *and* $f_k \to f$ *in* $L^1_{\mathrm{loc}}(\Omega)$. *Then*

$$\|Df\|(\Omega) \leq \liminf_{k\to+\infty} \|Df_k\|(\Omega).$$

Proof. Let $\phi \in \mathcal{C}^1_c(\Omega;\mathbb{R}^n)$ be such that $|\phi| \leq 1$. Then by the Structure Theorem (cf. (4.1.1)) and the fact that $|\phi| \leq 1$,

$$\begin{aligned}\int_\Omega f \operatorname{div}\phi \, d\mathcal{L}^n(x) &= \lim_{k\to+\infty}\int_\Omega f_k \operatorname{div}\phi \, d\mathcal{L}^n(x)\\ &= -\lim_{k\to+\infty}\int_\Omega \phi\cdot\sigma_k \, d\|Df_k\|\\ &\leq \liminf_{k\to+\infty}\|Df_k\|(\Omega).\end{aligned}$$

Hence, taking the supremum over all $\phi \in \mathcal{C}^1_c(\Omega;\mathbb{R}^n)$ with $|\phi| \leq 1$, we obtain

$$\|Df(\Omega)\| \leq \liminf_{k\to+\infty}\|Df_k\|(\Omega),$$

as required. The proof is complete. □

. . . .

4.2.2. *Approximation by Smooth Functions.*

t5.2-2 **Theorem 4.2.2** (Local Approximation by Smooth Functions). *Let* $f \in BV(\Omega)$. *Then there exist functions* $\{f_k\}_{k=1}^{+\infty} \subset BV(\Omega)\cap C^\infty(\Omega)$ *such that*

(i) $f_k \to f$ *in* $L^1(\Omega)$;
(ii) $\|Df_k\|(\Omega) \to \|Df\|(\Omega)$ *as* $k \to +\infty$.

Remark. *Note that in Theorem (4.2.2), we do not assume that* $\|D(f_k - f)\|(\Omega) \to 0$.

Proof.
(i). Fix $\epsilon > 0$. Given a positive integer m, define for each $k \in \mathbb{N}$ the open sets

$$U_k := \left\{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \frac{1}{m+k}\right\} \cap B(0,k+m).$$

Then choose $m \in \mathbb{N}$ so large so that

$$\|Df\|(\Omega\setminus U_1) < \epsilon. \tag{4.2.1}$$ {eq:5.2-1

Set $U_0 := \emptyset$ and define

$$V_k := U_{k+1}\setminus \overline{U}_{k-1}.$$

Let $\{\zeta_k\}_{k=1}^{+\infty}$ be a sequence of smooth functions such that

$$\begin{cases}\zeta_k \in \mathcal{C}^\infty_c(V_k), \quad 0 \leq \zeta_k \leq 1,\\ \displaystyle\sum_{k=1}^{+\infty}\zeta_k \equiv 1, \quad \text{on } \Omega.\end{cases}$$

We recall the standard mollifier $\eta : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $C > 0$ is chosen such that $\int_{\mathbb{R}^n} \eta \, d\mathcal{L}^n = 1$. We define then the sequence $\{\eta_\epsilon\}_{\epsilon>0}$ by

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right).$$

Now for each $k \in \mathbb{N}$, choose $\epsilon_k > 0$ so small that

$$\begin{cases} \operatorname{supp}(\eta_{\epsilon_k} * (f\zeta_k)) \subset V_k, \\ \displaystyle\int_\Omega |\eta_{\epsilon_k} * (f\zeta_k) - f\zeta_k| \, d\mathcal{L}^n < \frac{\epsilon}{2^k}, \\ \displaystyle\int_\Omega |\eta_{\epsilon_k} * (fD\zeta_k) - fD\zeta_k| \, d\mathcal{L}^n < \frac{\epsilon}{2^k}. \end{cases} \tag{4.2.2}$$

{eq:5.2-2

Define then

$$f_\epsilon := \sum_{k=1}^{+\infty} \eta_{\epsilon_k} * (f\zeta_k).$$

For each point $x \in \Omega$, there exists a neighborhood U_x such that there are only finitely many terms in this sum. Thus

$$f_\epsilon \in \mathcal{C}^\infty(\Omega).$$

(ii). Since also

$$f = \sum_{k=1}^{+\infty} f\zeta_k,$$

eq:5.2-2
(4.2.2) implies that

$$\|f_\epsilon - f\|_{L^1(\Omega)} \leq \sum_{k=1}^{+\infty} \int_\Omega |\eta_{\epsilon_k} * (f\zeta_k) - f\zeta_k| \, d\mathcal{L}^n < \epsilon.$$

Consequently

$$f_\epsilon \to f \quad \text{in } L^1(\Omega) \text{ as } \epsilon \to 0.$$

t5.2-1
(iii). According to Theorem (4.2.1),

$$\|Df\|(\Omega) \leq \liminf_{\epsilon\to 0} \|Df_\epsilon\|(\Omega). \tag{4.2.3}$$

{eq:5.2-3

(iv). Now let $\phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n)$, $|\phi| \leq 1$. Then

$$\begin{aligned} \int_\Omega f_\epsilon \operatorname{div}\phi \, d\mathcal{L}^n &= \sum_{k=1}^{+\infty} \int_\Omega \eta_{\epsilon_k} * (f\zeta_k) \operatorname{div}\phi \, d\mathcal{L}^n \\ &= \sum_{k=1}^{+\infty} \int_\Omega f\zeta_k \operatorname{div}(\eta_{\epsilon_k} * \phi) \, d\mathcal{L}^n \\ &= \sum_{k=1}^{+\infty} \int_\Omega f \operatorname{div}(\zeta_k(\eta_{\epsilon_k} * \phi)) \, d\mathcal{L}^n - \sum_{k=1}^{+\infty} \int_\Omega fD\zeta_k \cdot (\eta_{\epsilon_k} * \phi) \, d\mathcal{L}^n \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{+\infty} \int_\Omega f \operatorname{div}(\zeta_k(\eta_{\epsilon_k} * \phi))\, d\mathcal{L}^n - \sum_{k=1}^{+\infty} \int_\Omega \phi \cdot (\eta_{\epsilon_k} * (f D\zeta_k) - f D\zeta_k)\, d\mathcal{L}^n \\
&=: I_1^\epsilon + I_2^\epsilon.
\end{aligned}
$$

Here we have used the facts that $\operatorname{div}(\eta_{\epsilon_k} * \phi) = \eta_{\epsilon_k} * \operatorname{div} \phi$ and $\sum_{k=1}^{+\infty} D\zeta_k = 0$. Now $|\zeta_k(\eta_{\epsilon_k} * \phi)| \leq 1$ for each $k \in \mathbb{N}$ and each point $x \in \Omega$ belongs to at most three of the sets $\{V_k\}_{k=1}^{+\infty}$ by definition of V_k. Hence

$$
\begin{aligned}
|I_1^\epsilon| &= \left| \int_\Omega f \operatorname{div}(\zeta_1(\eta_{\epsilon_1} * \phi))\, d\mathcal{L}^n + \sum_{k=2}^{+\infty} \int_\Omega f \operatorname{div}(\zeta_k \eta_{\epsilon_k} * \phi)\, d\mathcal{L}^n \right| \\
&\leq \|Df\|(\Omega) + \sum_{k=2}^{+\infty} \|Df\|(V_k) \\
&\leq \|Df\|(\Omega) + 3\|Df\|(\Omega \setminus U_1) \\
&\leq \|Df\|(\Omega) + 3\epsilon,
\end{aligned}
$$

by (4.2.1). On the other hand, (4.2.2) implies that

$$
|I_2^\epsilon| < \epsilon.
$$

Therefore

$$
\int_\Omega f_\epsilon \operatorname{div} \phi\, d\mathcal{L}^n \leq \|Df\|(\Omega) + 4\epsilon,
$$

so evidently

$$
\|Df_\epsilon\|(\Omega) \leq \|Df\|(\Omega) + 4\epsilon.
$$

Finally, we have by (4.2.3)

$$
\|Df\|(\Omega) \leq \liminf_{\epsilon \to 0} \|Df_\epsilon\|(\Omega) \leq \liminf_{\epsilon \to 0} (\|Df\|(\Omega) + 4\epsilon) = \|Df(\Omega)\|.
$$

The proof is complete. □

t5.2-3 **Theorem 4.2.3** (Weak Approximation of Derivatives). *Let $f \in BV(\Omega)$, and let $\{f_k\}_{k=1}^{+\infty} \subset BV(\Omega) \cap \mathcal{C}^\infty(\Omega)$ be such that*

(i) $f_k \to f$ in $\mathcal{L}^1(\Omega)$;
(ii) $\|Df_k\|(\Omega) \to \|Df\|(\Omega)$ as $k \to +\infty$.

Define the vector–valued Radon measure

$$
\mu_k(B) := \int_{B \cap \Omega} Df_k\, d\mathcal{L}^n
$$

for each Borel set $B \subseteq \mathbb{R}^n$. Set also

$$
\mu(B) := \int_{B \cap \Omega} d[Df].
$$

Then

$$
\mu_k \rightharpoonup \mu
$$

weakly in the sense of vector–valued Radon measures on $\mathbb{R}^n$.

Remark. *Note that the existence of the sequence $\{f_k\}_{k=1}^{+\infty} \subset BV(\Omega) \cap \mathcal{C}^\infty(\Omega)$ satisfying assumptions (i) and (ii) is guaranteed by Theorem (4.2.2). Also recall that weak convergence here means*

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \phi \, d\mu_k = \int_{\mathbb{R}^n} \phi \, d\mu$$

for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. Fix $\phi \in \mathcal{C}_c^1(\mathbb{R}^n;\mathbb{R}^n)$ and $\epsilon > 0$. Choose $m \in \mathbb{N}$ so large that

$$U := \left\{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{m} \right\} \cap B(0,m)$$

satisfies

$$\|Df\|(\Omega \setminus U) < \epsilon.$$

Note also that $U \subset\subset \Omega$. Choose then a smooth cutoff function $\zeta : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\begin{cases} \zeta \equiv 1 \text{ on } U, \quad \operatorname{supp} \zeta \subset \Omega, \\ 0 \le \zeta \le 1. \end{cases}$$

Observe that

$$\begin{aligned} \int_{\mathbb{R}}^n \phi \, d\mu_k &= \int_\Omega \phi \cdot Df_k \, d\mathcal{L}^n \\ &= \int_\Omega \zeta\phi \cdot Df_k \, d\mathcal{L}^n + \int_\Omega (1-\zeta)\phi \cdot Df_k \, d\mathcal{L}^n \\ &= -\int_\Omega \operatorname{div}(\zeta\phi) f_k \, d\mathcal{L}^n + \int_\Omega (1-\zeta)\phi \cdot Df_k \, d\mathcal{L}^n, \end{aligned} \tag{4.2.4}$$

where we have used integration by parts on the first term in (4.2.4). Since $f_k \to f$ in $L^1(\Omega)$, we have by the Structure Theorem that the first term in (4.2.4) converges to

$$\begin{aligned} -\int_\Omega \operatorname{div}(\zeta\phi) f \, d\mathcal{L}^n &= \int_\Omega \zeta\phi \cdot d[Df] \\ &= \int_\Omega \phi \cdot d[Df] + \int_\Omega (\zeta - 1)\phi \cdot d[Df]. \end{aligned} \tag{4.2.5}$$

The second term in (4.2.5) is estimated by

$$\|\phi\|_{L^\infty(\Omega)} \|Df\|(\Omega \setminus U) \le C\epsilon.$$

Using the fact that $\|Df_k\|(\Omega) \to \|Df\|(\Omega)$ as $k \to +\infty$, we see that for k large enough, the second term in (4.2.4) may be estimated by

$$\|\phi\|_{L^\infty(\Omega)} \|Df_k\|(\Omega \setminus U) \le C\epsilon.$$

Hence

$$\left| \int_{\mathbb{R}^n} \phi \, d\mu_k - \int_{\mathbb{R}^n} \phi \, d\mu \right| \le C\epsilon$$

for all $k \in \mathbb{N}$ large enough. The proof is complete. □

. . . .

4.2.3. *Compactness.*

t5.2-4 **Theorem 4.2.4.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with $\partial\Omega$ Lipschitz. Assume that $\{f_k\}_{k=1}^{+\infty} \subset BV(\Omega)$ satisfies*

$$\sup_{k\in\mathbb{N}} \|f_k\|_{BV(\Omega)} < +\infty.$$

Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{+\infty}$ and a function $f \in BV(\Omega)$ such that

$$f_{k_j} \to f \quad in\ L^1(\Omega)$$

as $j \to +\infty$.

Proof. For $k \in \mathbb{N}$, choose by Theorem (4.2.2) functions $g_k \in \mathcal{C}^\infty(\Omega)$ so that

$$\begin{cases} \displaystyle\int_\Omega |f_k - g_k|\, d\mathcal{L}^n < \frac{1}{k}, \\ \sup_{k\in\mathbb{N}} \displaystyle\int_\Omega |Dg_k|\, d\mathcal{L}^n < +\infty. \end{cases} \tag{4.2.6}$$ {eq:5.2-6

By the Rellich–Kondrachov embedding theorem and the fact that Ω is bounded, there exists $f \in L^1(\Omega)$ and a subsequence $\{g_{k_j}\}_{j=1}^{+\infty}$ such that $g_{k_j} \to f$ in $L^1(\Omega)$ as $j \to +\infty$. But then (4.2.6) implies also that $f_{k_j} \to f$ in $L^1(\Omega)$. Thus by Theorem (4.2.1),

$$\|Df\|(\Omega) \le \liminf_{\epsilon\to 0} \|Df_k\|(\Omega) < +\infty,$$

so that $f \in BV(\Omega)$. The proof is complete. $\square$

4.3. **Traces.** We assume in this section that $\Omega \subset \mathbb{R}^n$ is open and bounded, with $\partial\Omega$ Lipschitz. Recall that since $\partial\Omega$ is Lipschitz, the outer unit normal ν exists $\mathcal{H}^{n-1}$−a.e. on $\partial\Omega$ by Rademacher's Theorem.

In this section we extend the notion of the trace operator to BV functions.

t5.3-1 **Theorem 4.3.1.** *Assume that $\Omega \subset \mathbb{R}^n$ is open and bounded, with $\partial\Omega$ Lipschitz. Then there exists a bounded linear operator*

$$T : BV(\Omega) \to L^1(\partial\Omega; \mathcal{H}^{n-1})$$

such that

$$\int_\Omega f \operatorname{div}\phi\, d\mathcal{L}^n = -\int_\Omega \phi \cdot\ d[Df] + \int_{\partial\Omega} (\phi\cdot\nu) Tf\, d\mathcal{H}^{n-1} \tag{4.3.1}$$ {eq:5.3-1

for all $f \in BV(\Omega)$ and $\phi \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}^n)$.

The point is that we now do not require ϕ to vanish near $\partial\Omega$.

Definition (Trace)**.** *The function Tf, which is uniquely defined up to sets of $\mathcal{H}^{n-1} \llcorner \partial\Omega$ measure zero, is called the* **trace** *of f on $\partial\Omega$.*

We interpret Tf as the "boundary values" of f on $\partial\Omega$.

Remark. *If $f \in W^{1,1}(\Omega) \subset BV(\Omega)$, then the definition of trace above and the definition of trace for Sobolev functions coincide.*

Proof.

i. We first introduce some notation:

(i) For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write $x = (x', x_n)$ for $x' := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. Similarly we write $y = (y', y_n)$.

(ii) For any $x \in \mathbb{R}^n$ and $r, h > 0$, define the open cylinder

$$C(x, r, h) := \{y \in \mathbb{R}^n : |y' - x'| < r, |y_n - x_n| < h\}.$$

Now since $\partial\Omega$ is Lipschitz, for each point $x \in \partial\Omega$ there exist $r, h > 0$ and a Lipschitz function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\max_{|x'-y'| \le r} |\gamma(y') - x_n| \le \frac{h}{4}$$

and, upon rotating and relabeling the coordinate axes if necessary,

$$\Omega \cap C(x, r, h) = \{y \in \mathbb{R}^n : |x' - y'| < r, \gamma(y') < y_n < x_n + h\}.$$

ii. Assume now temporarily that $f \in BV(\Omega) \cap \mathcal{C}^\infty(\Omega)$. Choose $x \in \partial\Omega$ and $r, h, \gamma, C(x, r, h)$ as above. Write

$$C := C(x, r, h).$$

Choose also $0 < \epsilon < \frac{h}{2}$ and $y \in \partial\Omega \cap C$, and define

$$f_\epsilon(y) := f(y', \gamma(y') + \epsilon).$$

Set also

$$C_{\delta,\epsilon} := \{y \in C : \gamma(y') + \delta < y_n < \gamma(y') + \epsilon\}$$

for $0 \le \delta < \epsilon < \frac{h}{2}$, and define $C_\epsilon := C_{0,\epsilon}$. Write $C^\epsilon := (C \cap \Omega) \setminus C_\epsilon$. Then

$$\begin{aligned} |f_\delta(y) - f_\epsilon(y)| &\le \int_\delta^\epsilon \left| \frac{\partial f}{\partial x_n}(y', \gamma(y') + t) \right| dt \\ &\le \int_\delta^\epsilon |Df(y', \gamma(y') + t)|\, dt, \end{aligned}$$

and consequently, since γ is Lipschitz, the area formula (cf. (3.3.1)) implies

$$\int_{\partial\Omega\cap C} |f_\delta - f_\epsilon|\, d\mathcal{H}^{n-1} \le C \int_{C_{\delta,\epsilon}} |Df|\, d\mathcal{L}^n(y) = C\|Df\|(C_{\delta,\epsilon}).$$

Therefore $\{f_\epsilon\}_{\epsilon>0}$ is Cauchy in $L^1(\partial\Omega \cap C; \mathcal{H}^{n-1})$, and thus the limit

$$Tf := \lim_{\epsilon\to 0} f_\epsilon$$

exists in $L^1(\partial\Omega \cap C; \mathcal{H}^{n-1})$. Furthermore, passing to the limit as $\delta \to 0$ in the previous inequality gives by Lebesgue's Dominated Convergence Theorem

$$\int_{\partial\Omega\cap C} |Tf - f_\epsilon|\, d\mathcal{H}^{n-1} \le C\|Df\|(C_\epsilon). \tag{4.3.2}$$

Next fix $\phi \in \mathcal{C}^1_c(C; \mathbb{R}^n)$. Then by the divergence theorem

$$\int_{C_\epsilon} f \operatorname{div}\phi\, d\mathcal{L}^n = -\int_{C_\epsilon} \phi \cdot Df\, d\mathcal{L}^n + \int_{\partial\Omega\cap C} f_\epsilon \phi_\epsilon \cdot \nu\, d\mathcal{H}^{n-1}.$$

Now sending $\epsilon \to 0$, we obtain

$$\int_{\Omega\cap C} f \operatorname{div}\phi\, d\mathcal{L}^n = -\int_{\Omega\cap C} \phi \cdot \sigma\, d\|Df\| + \int_{\partial\Omega\cap C} Tf\phi \cdot \nu\, d\mathcal{H}^{n-1} \tag{4.3.3}$$

by the above and the structure theorem (cf. (4.1.1)).

iii. Since $\partial\Omega$ is compact, we can cover $\partial\Omega$ with finitely many cylinders $C_i = C(x_i, r_i, h_i)$, $i = 1, \dots, N$, for which assertions (4.3.2) and (4.3.3) hold. An argument using a partition of unity subordinate to each of the terms in $\{C_i\}_{i=1}^N$ establishes (4.3.1). Observe also that (4.3.3) shows the definition of Tf to be the same up to sets of $\mathcal{H}^{n-1} \llcorner \partial\Omega$ on any part of $\partial\Omega$ that lies in two or more of the cylinders C_i.

iv. Assume now only that $f \in BV(\Omega)$. In this case, choose $f_k \in BV(\Omega) \cap \mathcal{C}^\infty(\Omega)$, $k = 1, 2, \dots$, such that

$$f_k \to f \quad \text{in } L^1(\Omega), \quad \|Df_k\|(\Omega) \to \|Df\|(\Omega),$$

and

$$\mu_k \to \mu \quad \text{weakly},$$

where the measures $\{\mu_k\}_{k=1}^{+\infty}$ and μ are defined as in Theorem (4.2.3). Recall that Theorem (4.2.3) also implies the existence of the approximating sequence $\{f_k\}_{k=1}^{+\infty}$ as well as the measures $\{\|Df_k\|\}_{k=1}^{+\infty}$.

v. We claim that $\{Tf_k\}_{k=1}^{+\infty}$ is a Cauchy sequence in $L^1(\partial\Omega; \mathcal{H}^{n-1})$.

To establish this, choose a cylinder $C = C(x, r, h)$. Fix $\epsilon > 0$, $y \in \partial\Omega \cap C$, and then define

$$f_k^\epsilon(y) := \frac{1}{\epsilon} \int_0^\epsilon f_k(y', \gamma(y') + t)\, dt = \frac{1}{\epsilon} \int_0^\epsilon (f_k)_t(y)\, dt.$$

Then (4.3.2) implies that

$$\begin{aligned}\int_{\partial\Omega \cap C} |Tf_k - f_k^\epsilon|\, d\mathcal{H}^{n-1} &\le \frac{1}{\epsilon} \int_0^\epsilon \int_{\partial\Omega \cap C} |Tf_k - (f_k)_t|\, d\mathcal{H}^{n-1}\, dt \\ &\le C\|Df_k\|(C_\epsilon).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\partial\Omega \cap C} |Tf_k - Tf_l|\, d\mathcal{H}^{n-1} \\ &\qquad \le \int_{\partial\Omega \cap C} |Tf_k - f_k^\epsilon|\, d\mathcal{H}^{n-1} + \int_{\partial\Omega \cap C} |Tf_l - f_l^\epsilon|\, d\mathcal{H}^{n-1} + \int_{\partial\Omega \cap C} |f_k^\epsilon - f_l^\epsilon|\, d\mathcal{H}^{n-1} \\ &\qquad \le C(\|Df_k\| + \|Df_l\|)(C_\epsilon) + \frac{C}{\epsilon} \int_{C_\epsilon} |f_k - f_l|\, d\mathcal{L}^n,\end{aligned}$$

and thus

$$\limsup_{k,l \to +\infty} \int_{\partial\Omega \cap C} |Tf_k - Tf_l|\, d\mathcal{H}^{n-1} \le C\|Df\|(\overline{C_\epsilon} \cap \Omega).$$

Since the RHS tends to zero as $\epsilon \to 0$, the claim is proved.

vi. By the claim in (v) and the fact that $L^1(\partial\Omega; \mathcal{H}^{n-1})$ is a Banach space, we may define

$$Tf := \lim_{k \to +\infty} Tf_k.$$

Note in particular that this definition does not depend on the choice of approximating sequence $\{f_k\}_{k=1}^{+\infty} \subseteq BV(\Omega) \cap \mathcal{C}^\infty(\Omega)$.

Finally, since (4.3.1) holds for each f_k, $k = 1, 2, \dots$, (4.3.1) holds also for f in the limit. The proof is complete. $\square$

t5.3-2 **Theorem 4.3.2.** *Assume that $\Omega \subset \mathbb{R}^n$ is open and bounded, with $\partial\Omega$ Lipschitz. Suppose also that $f \in BV(\Omega)$. Then for $\mathcal{H}^{n-1}$−a.e. $x \in \partial\Omega$,*

$$\lim_{r\to 0} \fint_{B(x,r)\cap\Omega} |f - Tf(x)| \, d\mathcal{L}^n = 0,$$

and so

$$Tf(x) = \lim_{r\to 0} \fint_{B(x,r)\cap\Omega} f(y) \, d\mathcal{L}^n(y).$$

Remark. *In particular, if $f \in BV(\Omega) \cap \mathcal{C}(\overline{\Omega})$, then*

$$Tf = f|_{\partial\Omega} \quad \mathcal{H}^{n-1} - a.e.$$

This justifies our interpretation of the trace of f as the boundary values of f on $\partial\Omega$.

Proof.
i. We first claim that for $\mathcal{H}^{n-1}$−a.e. $x \in \partial\Omega$,

$$\lim_{r\to 0} \frac{\|Df\|(B(x,r)\cap\Omega)}{r^{n-1}} = 0.$$

To see this, fix $\gamma > 0, \delta > \epsilon > 0$, and let

$$A_\gamma := \left\{ x \in \partial\Omega : \limsup_{r\to 0} \frac{\|Df\|(B(x,r)\cap\Omega)}{r^{n-1}} > \gamma \right\}.$$

Then for each $x \in A_\gamma$, there exists $0 < r < \epsilon$ such that

$$\frac{\|Df\|(B(x,r)\cap\Omega)}{r^{n-1}} \geq \gamma. \tag{4.3.4}$$ {eq:5.3-4

By the Vitali Covering Lemma, we obtain a countable collection of disjoint balls $\{B(x_i,r_i)\}_{i=1}^{+\infty}$ satisfying (4.3.4) eq:5.3-4 such that

$$A_\gamma \subset \bigcup_{i=1}^{+\infty} B(x_i, 5r_i).$$

Then

$$\begin{aligned}
\mathcal{H}^{n-1}_{10\delta}(A_\gamma) &\leq \sum_{i=1}^{+\infty} \alpha(n-1)(5r_i)^{n-1} \\
&\leq \frac{C}{\gamma} \sum_{i=1}^{+\infty} \|Df\|(B(x_i,r_i)\cap\Omega) \\
&\leq C\|Df\|(\Omega_\epsilon),
\end{aligned}$$

where

$$\Omega_\epsilon := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \epsilon\}.$$

Sending $\epsilon \to 0$, we find that $\mathcal{H}^{n-1}_{10\delta}(A_\gamma) = 0$ for all $\delta > 0$. This proves i.
ii. Now fix a point $x \in \partial\Omega$ such that

$$\lim_{r\to 0} \frac{\|Df\|(B(x,r)\cap\Omega)}{r^{n-1}} = 0$$

and

$$\lim_{r\to 0} \fint_{B(x,r)\cap\Omega} |Tf - Tf(x)| \, d\mathcal{H}^{n-1} = 0.$$

By (i) and the Lebesgue Differentiation Theorem, the above holds for $\mathcal{H}^{n-1}$–a.e. $x \in \partial\Omega$. Let $h = h(r) := 2\max\{1, 4\,\mathrm{Lip}(\gamma)\}r$, and consider the cylinders

$$C(r) := C(x, r, h).$$

Note that for $r > 0$ small enough, the cylinders $C(r)$ work in place of the cylinder C in the proof of Theorem (4.3.1). Thus estimates similar to those in the previous proof show that

$$\int_{\partial\Omega \cap C(r)} |Tf - f_\epsilon| \, d\mathcal{H}^{n-1} \le C\|Df\|(C(r) \cap \Omega),$$

where

$$f_\epsilon(y) := f(y', \gamma(y') + \epsilon), \quad y \in C(r) \cap \partial\Omega, \quad 0 < \epsilon < \frac{h(r)}{2}.$$

Thus, by the Coarea Formula (cf. (3.4.1)), we may estimate

$$\int_{B(x,r)\cap\Omega} |Tf(y', \gamma(y')) - f(y)| \, d\mathcal{L}^n(y) \le Cr\|Df\|(C(r) \cap \Omega).$$

Hence, we find

$$\begin{aligned}
\fint_{B(x,r)\cap\Omega} |f(y) - Tf(x)| \, d\mathcal{L}^n(y) &\le \frac{C}{r^{n-1}} \int_{C(r)\cap\partial\Omega} |Tf - Tf(x)| \, d\mathcal{H}^{n-1}(x) + \\
&\qquad \frac{C}{r^n} \int_{B(x,r)\cap\Omega} |Tf(y', \gamma(y')) - f(y)| \, d\mathcal{L}^n(y) \\
&\le o(1) + \frac{C}{r^{n-1}} \|Df\|(C(r) \cap \Omega) \\
&= o(1) \quad \text{as } \epsilon \to 0,
\end{aligned}$$

where the RHS follows from (i). The proof is complete. □

4.4. **Extensions.** In this section we state and prove a theorem that gives conditions upon which we may extend a BV function on a bounded open domain $\Omega \subset \mathbb{R}^n$ to all of $\mathbb{R}^n$.

t5.4-1 **Theorem 4.4.1.** *Assume that $\Omega \subset \mathbb{R}^n$ is open and bounded, with $\partial\Omega$ Lipschitz. Let $f_1 \in BV(\Omega)$, $f_2 \in BV(\mathbb{R}^n \setminus \Omega)$. Define*

$$\overline{f}(x) := \begin{cases} f_1(x), & x \in \Omega, \\ f_2(x), & x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Then

$$\overline{f} \in BV(\mathbb{R}^n),$$

and

$$\|D\overline{f}\|(\mathbb{R}^n) = \|Df_1\|(\Omega) + \|Df_2\|(\mathbb{R}^n \setminus \overline{\Omega}) + \int_{\partial\Omega} |Tf_1 - Tf_2| \, d\mathcal{H}^{n-1}.$$

Remark. *In particular, under the stated assumptions on Ω,*

(i) Clearly the extension

$$Ef := \begin{cases} f & \text{on } \Omega, \\ 0 & \text{on } \mathbb{R}^n \setminus \Omega \end{cases}$$

belongs to $BV(\mathbb{R}^n)$ if $f \in BV(\Omega)$, *(ii) The set Ω has finite perimeter and $\|\partial\Omega\|(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial\Omega)$.*

Proof.
i. Let $\phi \in \mathcal{C}^1_c(\mathbb{R}^n;\mathbb{R}^n)$, $|\phi| \le 1$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} \overline{f} \operatorname{div}\phi \, d\mathcal{L}^n &= \int_\Omega f_1 \operatorname{div}\phi\, d\mathcal{L}^n + \int_{\mathbb{R}^n\setminus\overline{\Omega}} f_2 \operatorname{div}\phi\, d\mathcal{L}^n \\ &= -\int_\Omega \phi\cdot d[Df_1] - \int_{\mathbb{R}^n\setminus\overline{\Omega}} \phi\cdot d[Df_2] + \\ &\qquad \int_{\partial\Omega} (Tf_1 - Tf_2)\phi\cdot\nu\, d\mathcal{H}^{n-1} \\ &\le \|Df_1\|(\Omega) + \|Df_2\|(\mathbb{R}^n\setminus\overline{\Omega}) + \int_{\partial\Omega} |Tf_1 - Tf_2|\, d\mathcal{H}^{n-1}.\end{aligned}$$

Thus $\overline{f} \in BV(\Omega)$, and

$$\|D\overline{f}\|(\mathbb{R}^n) \le \|Df_1\|(\Omega) + \|Df_2\|(\mathbb{R}^n\setminus\overline{\Omega}) + \int_{\partial\Omega} |Tf_1 - Tf_2|\, d\mathcal{H}^{n-1}.$$

ii. To show equality, observe that

$$-\int_{\mathbb{R}^n} \phi \cdot d[D\overline{f}] = -\int_\Omega \phi\cdot d[Df_1] - \int_{\mathbb{R}^n\setminus\overline{\Omega}} \phi\cdot d[Df_2] + \int_{\partial\Omega} (Tf_1 - Tf_2)\phi\cdot\nu\, d\mathcal{H}^{n-1} \quad (4.4.1)$$ {eq:5.4-1

for all $\phi \in \mathcal{C}^1_c(\mathbb{R}^n;\mathbb{R}^n)$. Thus

$$[D\overline{f}] = \begin{cases} [Df_1] & \text{on } \Omega, \\ [Df_2] & \text{on } \mathbb{R}^n\setminus\overline{\Omega}. \end{cases}$$

Consequently, (eq:5.4-1 4.4.1) implies that

$$-\int_{\partial\Omega} \phi\cdot d[D\overline{f}] = \int_{\partial\Omega} (Tf_1 - Tf_2)\phi\cdot\nu\, d\mathcal{H}^{n-1},$$

and hence

$$\|D\overline{f}\|(\partial\Omega) = \int_{\partial\Omega} |Tf_1 - Tf_2|\, d\mathcal{H}^{n-1},$$

as required. The proof is complete. □

4.5. **Coarea Formula for BV Functions.** We want to relate the variation measure of f and the perimeters of its level sets.

Remark (Notation)**.** *For $f : \Omega \to \mathbb{R}$ and $t \in \mathbb{R}$, we define*

$$E_t := \{x \in \Omega : f(x) > t\}.$$

l5.5-1 **Lemma 4.5.1.** *If $f \in BV(\Omega)$, the mapping*

$$t \mapsto \|\partial E_t\|(\Omega), \quad t \in \mathbb{R},$$

in $\mathcal{L}^1$−measurable.

Proof. The mapping

$$(x,t) \mapsto \mathbb{1}_{E_t}(x)$$

is $(\mathcal{L}^n \times \mathcal{L}^1)-$measurable by the Fubini–Tonelli Theorem, and thus, for each $\phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n)$, the function

$$t \mapsto \int_{E_t} \operatorname{div}\phi \, d\mathcal{L}^n = \int_\Omega \mathbb{1}_{E_t} \operatorname{div}\phi \, d\mathcal{L}^n$$

is $\mathcal{L}^1-$measurable. Let D denote any countable dense subset of $\mathcal{C}_c^1(\Omega;\mathbb{R}^n)$. Then

$$t \mapsto \|\partial E_t\|(\Omega) = \sup_{\substack{\phi\in D\\ |\phi|\le 1}} \int_{E_t} \operatorname{div}\phi \, d\mathcal{L}^n$$

is $\mathcal{L}^1-$measurable. The proof is complete. □

t5.5-1 **Theorem 4.5.1** (Coarea Formula for BV Functions)**.** *Let $f \in BV(\Omega)$. Then*

(i) E_t has finite perimeter for $\mathcal{L}^1-$a.e. $t \in \mathbb{R}$, and

$$\|Df\|(\Omega) = \int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega) \, d\mathcal{L}^1(t).$$

(ii) Conversely, if $f \in L^1(\Omega)$ and

$$\int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega) \, d\mathcal{L}^1(t) < +\infty,$$

then $f \in BV(\Omega)$.

Remark. *Compare Theorem (4.5.1) with Proposition (3.4.2).*

Proof. Let $\phi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n)$, $|\phi| \le 1$.

i. We first claim that $\int_\Omega f \operatorname{div}\phi \, d\mathcal{L}^n = \int_{-\infty}^{+\infty}\left(\int_{E_t} \operatorname{div}\phi \, d\mathcal{L}^n\right) \, d\mathcal{L}^1(t)$.

To see this, first suppose that $f \ge 0$, so that the Rising Sun Lemma gives

$$f(x) = \int_0^{+\infty} \mathbb{1}_{E_t}(x) \, d\mathcal{L}^1(t)$$

for a.e. $x \in \Omega$. Thus

$$\begin{aligned}
\int_\Omega f \operatorname{div}\phi \, d\mathcal{L}^n &= \int_\Omega \left(\int_0^{+\infty} \mathbb{1}_{E_t}(x) \, d\mathcal{L}^1(t)\right) \operatorname{div}\phi(x) \, d\mathcal{L}^n(x) \\
&= \int_0^{+\infty} \left(\int_\Omega \mathbb{1}_{E_t}(x) \operatorname{div}\phi(x) \, d\mathcal{L}^n(x)\right) \, d\mathcal{L}^1(t) \\
&= \int_0^{+\infty} \left(\int_{E_t} \operatorname{div}\phi \, d\mathcal{L}^n(x)\right) \, d\mathcal{L}^1(t).
\end{aligned}$$

Similarly, if $f \le 0$, then

$$f(x) = \int_{-\infty}^0 (\mathbb{1}_{E_t}(x) - 1) \, d\mathcal{L}^1(t),$$

so that

$$\int_\Omega f \operatorname{div}\phi \, d\mathcal{L}^n(x) = \int_\Omega \left(\int_{-\infty}^0 (\mathbb{1}_{E_t}(x) - 1) \, d\mathcal{L}^1(t)\right) \operatorname{div}\phi(x) \, d\mathcal{L}^n(x)$$

$$= \int_{-\infty}^{0} \left(\int_{\Omega} (\mathbb{1}_{E_t}(x) - 1) \operatorname{div} \phi(x) \, d\mathcal{L}^n(x) \right) \, d\mathcal{L}^1(t)$$
$$= \int_{-\infty}^{0} \left(\int_{E_t} \operatorname{div} \phi \, d\mathcal{L}^n(x) \right) \, d\mathcal{L}^1(t).$$

For the general case, write $f = f^+ - f^-$. This proves the claim i.

ii. From (i) we see that for all $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$, $|\phi| \leq 1$,

$$\int_{\Omega} f \operatorname{div} \phi \, d\mathcal{L}^n(x) \leq \int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega) \, d\mathcal{L}^1(t).$$

Hence, taking the supremum over all such ϕ on the LHS, we obtain

$$\|Df\|(\Omega) \leq \int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega) \, d\mathcal{L}^1(t). \tag{4.5.1}$$
{eq:5.5-1

iii. We next claim that

$$\|Df\|(\Omega) = \int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega) \, d\mathcal{L}^1(t)$$

for all $f \in BV(\Omega) \cap \mathcal{C}^\infty(\Omega)$.

Let

$$m(t) := \int_{\Omega \setminus E_t} |Df| \, d\mathcal{L}^n = \int_{\{f \leq t\}} |Df| \, d\mathcal{L}^n.$$

Then the function m is nondecreasing, and thus m' exists $\mathcal{L}^1$−a.e., with

$$\int_{-\infty}^{+\infty} m'(t) \, d\mathcal{L}^1(t) \leq \int_{\Omega} |Df| \, d\mathcal{L}^n. \tag{4.5.2}$$
{eq:5.5-2

Next fix any $-\infty < t < +\infty$, $r > 0$, and define $\eta : \mathbb{R} \to \mathbb{R}$ as follows:

$$\eta(s) := \begin{cases} 0, & s \leq t, \\ \frac{s-t}{r}, & t \leq s \leq t + r, \\ 1, & s \geq t + r. \end{cases}$$

Then

$$\eta'(s) = \begin{cases} \frac{1}{r}, & t < s < t + r, \\ 0, & s < t \text{ or } s > t + r. \end{cases}$$

Thus, for all $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$, we have

$$-\int_{\Omega} \eta(f(x)) \operatorname{div} \phi(x) \, d\mathcal{L}^n(x) = \int_{\Omega} \eta'(f(x)) Df(x) \cdot \phi(x) \, d\mathcal{L}^n(x)$$
$$= \frac{1}{r} \int_{E_t \setminus E_{t+r}} Df \cdot \phi \, d\mathcal{L}^n. \tag{4.5.3}$$
{eq:5.5-3

Now observe that

$$\frac{m(t+r) - m(t)}{r} = \frac{1}{r} \left[\int_{\Omega \setminus E_{t+r}} |Df| \, d\mathcal{L}^n - \int_{\Omega \setminus E_t} |Df| \, d\mathcal{L}^n \right]$$
$$= \frac{1}{r} \int_{E_t \setminus E_{t+r}} |Df| \, d\mathcal{L}^n$$

$$\begin{aligned}
&\geq \frac{1}{r} \int_{E_t \setminus E_{t+r}} Df \cdot \phi \, d\mathcal{L}^n \\
&= - \int_{\Omega} \eta(f(x)) \operatorname{div} \phi \, d\mathcal{L}^n,
\end{aligned}$$

where the RHS follows from (4.5.3). For all $t \in \mathbb{R}$ such that $m'(t)$ exists, we let $r \to 0$ to find that

$$m'(t) \geq - \int_{E_t} \operatorname{div} \phi \, d\mathcal{L}^n$$

for $\mathcal{L}^1-$a.e. $t \in \mathbb{R}$. Taking the supremum over all $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$, $|\phi| \leq 1$, we have

$$\|\partial E_t\|(\Omega) \leq m'(t).$$

Recalling (4.5.2), we then find

$$\int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega) \, d\mathcal{L}^1(t) \leq \int_{\Omega} |Df| \, d\mathcal{L}^n = \|Df\|(\Omega).$$

In view of (4.5.1), this proves the claim in (iii).

iv. We now show that part (iii) holds for all $f \in BV(\Omega)$.

Fix $f \in BV(\Omega)$ and choose a sequence $\{f_k\}_{k=1}^{+\infty} \subset BV(\Omega) \cap \mathcal{C}^{\infty}(\Omega)$ as in Theorem (4.2.2). Recall that

$$f_k \to f \quad \text{in } L^1(\Omega) \text{ as } k \to +\infty.$$

Define

$$E_t^k := \{x \in \Omega : f_k(x) > t\}.$$

Now

$$\int_{-\infty}^{+\infty} |\mathbb{1}_{E_t^k}(x) - \mathbb{1}_{E_t}(x)| \, d\mathcal{L}^1(t) = \int_{\min\{f(x), f_k(x)\}}^{\max\{f(x), f_k(x)\}} d\mathcal{L}^1(t) = |f_k(x) - f(x)|.$$

Consequently,

$$\int_{\Omega} |f_k(x) - f(x)| \, d\mathcal{L}^n(x) = \int_{-\infty}^{+\infty} \left(\int_{\Omega} |\mathbb{1}_{E_t^k}(x) - \mathbb{1}_{E_t}(x)| \, d\mathcal{L}^n(x) \right) d\mathcal{L}^1(t).$$

Since $f_k \to f$ in $L^1(\Omega)$, there exists a subsequence which, reindexing by k if necessary, satisfies

$$\mathbb{1}_{E_t^k} \to \mathbb{1}_{E_t} \quad \text{in } L^1(\Omega), \quad \text{for } \mathcal{L}^1 - \text{a.e. } t \in \mathbb{R}.$$

Then, by the Lower Semicontinuity Theorem (cf. (4.2.1)),

$$\|\partial E_t\|(\Omega) \leq \liminf_{k \to +\infty} \|\partial E_t^k\|(\Omega).$$

Thus, Fatou's Lemma gives

$$\begin{aligned}
\int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega) \, d\mathcal{L}^1(t) &\overset{F.L.}{\leq} \liminf_{k \to +\infty} \int_{-\infty}^{+\infty} \|\partial E_t^k\|(\Omega) \, d\mathcal{L}^1(t) \\
&= \lim_{k \to +\infty} \|Df_k\|(\Omega) \\
&= \|Df\|(\Omega),
\end{aligned}$$

where the RHS follows from the conclusion of Theorem (4.2.2) In view of (4.5.1), this proves (i).

eq:5.5-1

v. Assertion (ii) of the theorem follows immediately from (4.5.1), noting that

$$\|f\|_{BV(\Omega)} \leq \|f\|_{L^1(\Omega)} + \|Df\|(\Omega) \leq \|f\|_{L^1(\Omega)} + \int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega)\, d\mathcal{L}^1(t) < +\infty.$$

The proof is complete. □

4.6. **Isoperimetric Inequalities.** In this section we develop certain inequalities relating the $\mathcal{L}^n$−measure of a set to its perimeter.

. . . .

4.6.1. *Sobolev and Poincaré's Inequalities for BV Functions.* We begin by stating a version of Sobolev– and Poincaré –type inequalities for BV functions. We first recall the GNS inequality from PDE:

Remark. *Assume that $1 \leq p < n$. Then there exists a constant $C > 0$, depending on only p and n, such that for all $u \in \mathcal{C}_c^1(\mathbb{R}^n)$,*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

where p^* denotes the Sobolev conjugate of p,

$$p^* = \frac{np}{n-p}, \quad 1 \leq p < n.$$

t5.6-1 **Theorem 4.6.1.**

(i) There exists a constant $C_1 > 0$ such that

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C_1\|Df\|(\mathbb{R}^n)$$

for all $f \in BV(\mathbb{R}^n)$;

(ii) There exists a constant $C_2 > 0$ such that

$$\|f - (f)_{x,r}\|_{L^{\frac{n}{n-1}}(B(x,r))} \leq C_2\|Df\|(U(x,r))$$

for all $B(x,r) \subset \mathbb{R}^n$, $f \in BV_{\text{loc}}(\mathbb{R}^n)$, where

$$(f)_{x,r} := \fint_{B(x,r)} f(y)\, d\mathcal{L}^n(y)$$

denotes the average of f over $B(x,r)$;

(iii) For each $0 < \alpha \leq 1$, there exists a constant $C_3 = C_3(\alpha) > 0$ such that

$$\|f\|_{L^{\frac{n}{n-1}}(B(x,r))} \leq C_3\|Df\|(U(x,r))$$

for all $B(x,r) \subset \mathbb{R}^n$ and all $f \in BV_{\text{loc}}(\mathbb{R}^n)$, satisfying

$$\frac{\mathcal{L}^n(B(x,r) \cap \{f = 0\})}{\mathcal{L}^n(B(x,r))} \geq \alpha.$$

Remark. *Notice that (i) is a GNS-type inequality for BV functions, with $p = 1$. Assertions (ii) and (iii) are Poincaré–type inequalities.*

Proof.

i. Choose a sequence $\{f_k\}_{k=1}^{+\infty} \subset \mathcal{C}_c^\infty(\mathbb{R}^n)$ so that

$$\begin{cases} f_k \to f \text{ in } L^1(\mathbb{R}^n), \quad f_k \to f \ \mathcal{L}^1 - \text{a.e.}, \\ \|Df_k\|(\mathbb{R}^n) \to \|Df\|(\mathbb{R}^n). \end{cases}$$

Then by Fatou's Lemma and the Gagliardo–Nirenberg–Sobolev inequality with $p = 1$,

$$\begin{aligned} \|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} &\overset{F.L.}{\leq} \liminf_{k\to+\infty} \|f_k\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \\ &\overset{G.N.S.}{\leq} \lim_{k\to+\infty} C_1\|Df_k\|_{L^1(\mathbb{R}^n)} \\ &= C_1\|Df\|(\mathbb{R}^n). \end{aligned}$$

This proves assertion (i).

ii. Statement (ii) follows similarly from the usual Poincaré inequality for Sobolev functions.

iii. Suppose that

$$\frac{\mathcal{L}^n(B(x,r) \cap \{f = 0\})}{\mathcal{L}^n(B(x,r))} \geq \alpha > 0. \tag{4.6.1}$$

{eq:5.6-1

Then by assertion (ii),

$$\begin{aligned} \|f\|_{L^{\frac{n}{n-1}}(B(x,r))} &\leq \|f - (f)_{x,r}\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} + \|(f)_{x,r}\|_{L^{\frac{n}{n-1}}(B(x,r))} \\ &\leq C_2\|Df\|(U(x,r)) + |(f)_{x,r}|(\mathcal{L}^n(B(x,r)))^{1-\frac{1}{n}}. \end{aligned} \tag{4.6.2}$$

{eq:5.6-2

But, by applying Hölder's inequality,

$$\begin{aligned} |(f)_{x,r}|(\mathcal{L}^n(B(x,r)))^{1-\frac{1}{n}} &\leq \frac{1}{\mathcal{L}^n(B(x,r))^{\frac{1}{n}}} \int_{B(x,r)\cap\{f\neq 0\}} |f| \, d\mathcal{L}^n(y) \\ &\leq \left(\int_{B(x,r)} |f|^{\frac{n}{n-1}} \, d\mathcal{L}^n(y) \right)^{1-\frac{1}{n}} \left(\frac{\mathcal{L}^n(B(x,r) \cap \{f \neq 0\})}{\mathcal{L}^n(B(x,r))} \right)^{\frac{1}{n}} \\ &\leq \|f\|_{L^{\frac{n}{n-1}}(B(x,r))} (1-\alpha)^{\frac{1}{n}}, \end{aligned}$$

where the RHS follows from (4.6.1). Employing this estimate in (4.6.2), we calculate

$$(1 - (1-\alpha)^{\frac{1}{n}})\|f\|_{L^{\frac{n}{n-1}}(B(x,r))} \leq C_2\|Df\|(U(x,r)),$$

so that

$$\|f\|_{L^{\frac{n}{n-1}}(B(x,r))} \leq \frac{C_2}{(1 - (1-\alpha)^{\frac{1}{n}})}\|Df\|(U(x,r)).$$

The proof is complete. $\square$

. . . .

4.6.2. *Isoperimetric Inequalities.*

t5.6-2 **Theorem 4.6.2.** *Let E be a bounded set of finite perimeter in $\mathbb{R}^n$. Then there exist constants $C_1, C_2 > 0$ such that*

(i) $\mathcal{L}^n(E)^{1-\frac{1}{n}} \leq C_1 \|\partial E\|(\mathbb{R}^n)$;

(ii) For each ball $B(x,r) \subset \mathbb{R}^n$,

$$\min\{\mathcal{L}^n(B(x,r)\cap E), \mathcal{L}^n(B(x,r)\setminus E)\}^{1-\frac{1}{n}} \leq 2C_2\|\partial E\|(U(x,r)).$$

Remark. *Assertion (i) of Theorem (4.6.2) is the* **isoperimetric inequality**, *and assertion (ii) is the* **relative isoperimetric inequality**. *Note that (i) states that the $\mathcal{L}^n-$measure of such a set E is bounded above by its perimeter measure.*

Proof.

i. Putting $f = \mathbb{1}_E$ in Theorem (4.6.1)(i) proves assertion (i).

ii. Next, let $f = \mathbb{1}_{B(x,r)\cap E}$ in Theorem (4.6.1)(ii) to obtain

$$(f)_{x,r} = \frac{\mathcal{L}^n(B(x,r)\cap E)}{\mathcal{L}^n(B(x,r))}.$$

Thus

$$\begin{aligned}\int_{B(x,r)} |f-(f)_{x,r}|^{\frac{n}{n-1}}\, d\mathcal{L}^n(y) &= \int_{B(x,r)\setminus E} |(f)_{x,r}|^{\frac{n}{n-1}}\, d\mathcal{L}^n(y) + \int_{B(x,r)\cap E} |1-(f)_{x,r}|^{\frac{n}{n-1}}\, d\mathcal{L}^n(y)\\ &= \left(\frac{\mathcal{L}^n(B(x,r)\cap E)}{\mathcal{L}^n(B(x,r))}\right)^{\frac{n}{n-1}} \mathcal{L}^n(B(x,r)\setminus E) + \left(\frac{\mathcal{L}^n(B(x,r)\setminus E)}{\mathcal{L}^n(B(x,r))}\right)^{\frac{n}{n-1}} \mathcal{L}^n(B(x,r)\cap E).\end{aligned}$$

Now if $\mathcal{L}^n(B(x,r)\cap E) \leq \mathcal{L}^n(B(x,r)\setminus E)$, then by the above,

$$\begin{aligned}\left(\int_{B(x,r)} |f-(f)_{x,r}|^{\frac{n}{n-1}}\, d\mathcal{L}^n(y)\right)^{1-\frac{1}{n}} &\geq \left(\left[\frac{\mathcal{L}^n(B(x,r)\setminus E)}{\mathcal{L}^n(B(x,r))}\right]^{\frac{n}{n-1}} \mathcal{L}^n(B(x,r)\cap E)\right)^{1-\frac{1}{n}}\\ &= \left[\frac{\mathcal{L}^n(B(x,r)\setminus E)}{\mathcal{L}^n(B(x,r))}\right] \mathcal{L}^n(B(x,r)\cap E)^{1-\frac{1}{n}}\\ &\geq \frac{1}{2}\mathcal{L}^n(B(x,r)\cap E)^{1-\frac{1}{n}}\\ &= \frac{1}{2}\min\{\mathcal{L}^n(B(x,r)\cap E), \mathcal{L}^n(B(x,r)\setminus E)\}^{1-\frac{1}{n}},\end{aligned}$$

where the RHS follows from the fact that $\mathcal{L}^n(B(x,r)\cap E) + \mathcal{L}^n(B(x,r)\setminus E) = \mathcal{L}^n(B(x,r))$ and $\mathcal{L}^n(B(x,r)\cap E) \leq \mathcal{L}^n(B(x,r)\setminus E)$, so that

$$\frac{\mathcal{L}^n(B(x,r)\setminus E)}{\mathcal{L}^n(B(x,r))} \geq \frac{1}{2}.$$

The proof is complete. □

Remark. *We have shown that the GNS inequality implies the isoperimetric inequality, as shown in the proof of Theorem (4.6.2)(i). In fact, the converse is true as well.*

To see this, assume that $f \in C_c^1(\mathbb{R}^n)$, $f \geq 0$. We calculate by Theorem (4.5.1)

$$\int_{\mathbb{R}^n} |Df|\, d\mathcal{L}^n(x) = \|Df\|(\mathbb{R}^n) = \int_{-\infty}^{+\infty} \|\partial E_t\|(\mathbb{R}^n)\, d\mathcal{L}^1(t)$$

$$
\begin{aligned}
&= \int_0^{+\infty} \|\partial E_t\|(\mathbb{R}^n)\, d\mathcal{L}^1(t) \\
&\geq \frac{1}{C_1} \int_0^{+\infty} \mathcal{L}^n(E_t)^{1-\frac{1}{n}}\, d\mathcal{L}^1(t),
\end{aligned}
$$

where the RHS follows from Theorem (4.6.2)(i). Now let

$$
f_t := \min\{t, f\}, \quad \chi(t) := \left(\int_{\mathbb{R}^n} f_t^{\frac{n}{n-1}}\, d\mathcal{L}^n(x) \right)^{1-\frac{1}{n}}, \quad t \in \mathbb{R}.
$$

Then χ is nondecreasing on $(0, +\infty)$, and

$$
\lim_{t \to +\infty} \chi(t) = \left(\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}}\, d\mathcal{L}^n(x) \right)^{1-\frac{1}{n}}.
$$

Also, for $h > 0$,

$$
\begin{aligned}
0 &\leq \chi(t+h) - \chi(t) \\
&\leq \left(\int_{\mathbb{R}^n} |f_{t+h} - f_t|^{\frac{n}{n-1}}\, d\mathcal{L}^n(x) \right)^{1-\frac{1}{n}} \\
&\leq h \mathcal{L}^n(E_t)^{1-\frac{1}{n}}.
\end{aligned}
$$

Thus χ is locally Lipschitz, and

$$
\chi'(t) \leq \mathcal{L}^n(E_t)^{1-\frac{1}{n}}
$$

for $\mathcal{L}^1$–a.e. $t \in (0, +\infty)$. Integrating χ' from 0 to $+\infty$ gives

$$
\begin{aligned}
\left(\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}}\, d\mathcal{L}^n(x) \right)^{1-\frac{1}{n}} &= \int_0^{+\infty} \chi'(t)\, d\mathcal{L}^1(t) \\
&\leq \int_0^{+\infty} \mathcal{L}^n(E_t)^{\frac{n}{n-1}}\, d\mathcal{L}^1(t) \\
&\leq C_1 \int_0^{+\infty} \|\partial E_t\|(\mathbb{R}^n)\, d\mathcal{L}^1(t) \\
&= C_1 \|Df\|(\mathbb{R}^n) \\
&= C_1 \int_{\mathbb{R}^n} |Df|\, d\mathcal{L}^n(x),
\end{aligned}
$$

where the RHS follows from the coarea formula (cf. Theorem (4.5.1)(i).)

. . . .

4.6.3. $\mathcal{H}^{n-1}$ *and* Cap_1. We first define the p–capacity of a set $A \subseteq \mathbb{R}^n$. Fix $1 \leq p < n$.

Definition (K^p)**.** *We define*

$$
K^p := \{f : \mathbb{R}^n \to \mathbb{R} : f \geq 0,\ f \in L^{p^*}(\mathbb{R}^n),\ Df \in L^p(\mathbb{R}^n; \mathbb{R}^n)\}.
$$

Definition (p–Capacity)**.** *For any $A \subseteq \mathbb{R}^n$, we define the p–capacity of A by*

$$
\mathrm{Cap}_p(A) := \inf \left\{ \int_{\mathbb{R}^n} |Df|^p\, d\mathcal{L}^n : f \in K^p,\ A \subset \{x \in \mathbb{R}^n : f(x) \geq 1\}^\circ \right\}.
$$

The following theorem is a first application of the isoperimetric inequalities.

t5.6-3 **Theorem 4.6.3.** *Assume that* $A \subset \mathbb{R}^n$ *is compact. Then* $\mathrm{Cap}_1(A) = 0$ *if and only if* $\mathcal{H}^{n-1}(A) = 0$.

4.7. **The Reduced Boundary.** We now more closely examine the structure of sets of locally finite perimeter. Recall from §4.1 that given an open set $\Omega \subset \mathbb{R}^n$, a Lebesgue–measurable set $E \subset \mathbb{R}^n$ is said to have locally finite perimeter in Ω if

$$\mathbb{1}_E \in BV_{\mathrm{loc}}(\Omega).$$

We verify that such a set has a $\mathcal{C}^1$ boundary in a measure theoretical sense.

. . . .

4.7.1. *Estimates.* We assume that

$$E \text{ is a set of locally finite perimeter in } \mathbb{R}^n.$$

Recall that for a set of locally finite perimeter, the Structure Theorem (cf. (4.1.1) t5.1-1) states that for all $\phi \in \mathcal{C}^1_c(\Omega; \mathbb{R}^n)$,

$$\int_E \operatorname{div} \phi \, d\mathcal{L}^n(x) = -\int_\Omega \phi \cdot \sigma \, d\mu$$

for some Radon measure μ on Ω and a μ–measurable function $\sigma : \Omega \to \mathbb{R}^n$ such that $|\sigma(x)| = 1$ μ–a.e. on Ω. Recall also that we write $\|\partial E\|$ for the perimeter measure μ and $\nu_E := -\sigma$.

Definition (Reduced Boundary). *Let* $x \in \mathbb{R}^n$. *We say that* $x \in \partial^* E$, *the* **reduced boundary** *of* E, *if the following three conditions hold:*

(i) $\|\partial E\|(B(x,r)) > 0$ *for all* $r > 0$;
(ii) $\lim_{r\to 0} \fint_{B(x,r)} \nu_E \, d\|\partial E\| = \nu_E(x)$;
(iii) $|\nu_E(x)| = 1$.

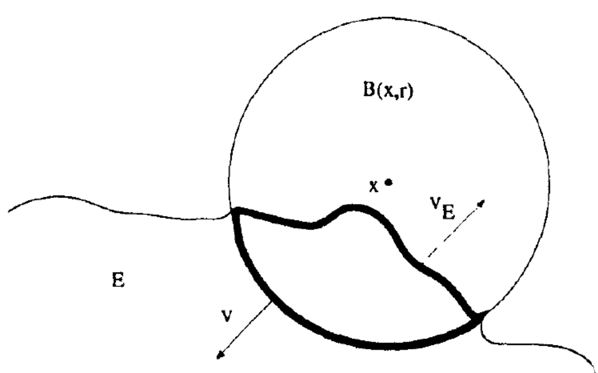

FIGURE 4.7.1. Normals to E and to $B(x,r)$.

We interpret ν as the outer unit normal to $B(x,r)$ (or Ω) and ν_E as the outer unit normal to E. We may also interpret the reduced boundary as a measure–theoretic smooth boundary – note that, given that E is a set of locally finite perimeter, then $\mathcal{H}^{n-1}$–a.e. point $x \in \partial E$ belongs to $\partial^* E$ and an outer unit normal to E exists at such points x.

Remark. *Notice that*

$$\|\partial E\|(\mathbb{R}^n \setminus \partial^* E) = 0.$$

l5.7-1 **Lemma 4.7.1.** *Let $\phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$. Then for each $x \in \mathbb{R}^n$,*

$$\int_{E\cap B(x,r)} \operatorname{div}\phi \, d\mathcal{L}^n = \int_{B(x,r)} \phi \cdot \nu_E \, d\|\partial E\| + \int_{E\cap \partial B(x,r)} \phi \cdot \nu \, d\mathcal{H}^{n-1}$$

for $\mathcal{L}^1$–a.e. $r > 0$, where ν denotes the outward unit normal to $\partial B(x,r)$.

Proof. Assume that $h : \mathbb{R}^n \to \mathbb{R}$ is smooth. Then by the product rule,

$$\int_E \operatorname{div}(h\phi)\, d\mathcal{L}^n = \int_E h \operatorname{div}\phi \, d\mathcal{L}^n + \int_E Dh \cdot \phi \, d\mathcal{L}^n.$$

Thus

$$\int_{\mathbb{R}^n} h\phi \cdot \nu_E \, d\|\partial E\| = \int_E h \operatorname{div}\phi \, d\mathcal{L}^n + \int_E Dh \cdot \phi \, d\mathcal{L}^n. \tag{4.7.1}$$ {eq:5.7-1

By approximation, (4.7.1) holds also for

$$h_\epsilon(y) := g_\epsilon(|y-x|),$$

where

$$g_\epsilon(s) := \begin{cases} 1, & 0 \le s \le r, \\ \frac{r-s+\epsilon}{\epsilon}, & r \le s \le r+\epsilon, \\ 0, & s \ge r+\epsilon. \end{cases}$$

Notice that

$$g_\epsilon'(s) = \begin{cases} 0, & 0 \le s < r \text{ or } s > r+\epsilon, \\ -\frac{1}{\epsilon}, & r < s < r+\epsilon, \end{cases}$$

and therefore

$$Dh_\epsilon(y) = \begin{cases} 0, & |y-x| < r \text{ or } |y-x| > r+\epsilon, \\ -\frac{1}{\epsilon}\frac{y-x}{|y-x|}, & r < |y-x| < r+\epsilon. \end{cases}$$

Setting $h = h_\epsilon$ in (4.7.1), we obtain

$$\int_{\mathbb{R}^n} h_\epsilon \phi \cdot \nu_E \, d\|\partial E\| = \int_E h_\epsilon \operatorname{div}\phi \, d\mathcal{L}^n - \frac{1}{\epsilon}\int_{E\cap\{y : r<|y-x|<r+\epsilon\}} \phi \cdot \frac{y-x}{|y-x|}\, d\mathcal{L}^n.$$

Letting $\epsilon \to 0$ and using polar coordinates (cf. Proposition (3.4.1)), we see that

$$\int_{B(x,r)} \phi \cdot \nu_E \, d\|\partial E\| = \int_{E\cap B(x,r)} \operatorname{div}\phi \, d\mathcal{L}^n - \int_{E\cap\partial B(x,r)} \phi \cdot \nu \, d\mathcal{H}^{n-1}$$

for $\mathcal{L}^1$–a.e. $r > 0$. ☐

l5.7-2 **Lemma 4.7.2.** *There exist positive constants $A_1, \dots, A_5$, depending only on n, such that for each $x \in \partial^* E$,*

(i) $\liminf_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap E)}{r^n} > A_1 > 0$,

(ii) $\liminf_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\setminus E)}{r^n} > A_2 > 0$,

(iii) $\liminf_{r\to 0} \frac{\|\partial E\|(B(x,r))}{r^{n-1}} > A_3 > 0$,
(iv) $\limsup_{r\to 0} \frac{\|\partial E\|(B(x,r))}{r^{n-1}} \le A_4$,
(v) $\limsup_{r\to 0} \frac{\|\partial (E\cap B(x,r))(\mathbb{R}^n)}{r^{n-1}} \le A_5$.

Proof.
i. Fix $x \in \partial^* E$. By Lemma (4.7.1), for $\mathcal{L}^1-$a.e. $r > 0$,

$$\|\partial(E\cap B(x,r))\|(\mathbb{R}^n) \le \|\partial E\|(B(x,r)) + \mathcal{H}^{n-1}(E\cap \partial B(x,r)). \tag{4.7.2}$$

On the other hand, choose $\phi \in \mathcal{C}_c^1(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$\phi \equiv \nu_E \quad \text{on } B(x,r).$$

Then by Lemma (4.7.1),

$$\int_{B(x,r)} \nu_E(x)\cdot \nu_E \, d\|\partial E\| = -\int_{E\cap B(x,r)} \nu_E(x)\cdot \nu \, d\mathcal{H}^{n-1}. \tag{4.7.3}$$

Since $x \in \partial^* E$, we have

$$\lim_{r\to 0} \nu_E(x)\cdot \fint_{B(x,r)} \nu_E \, d\|\partial E\| = |\nu_E(x)|^2 = 1.$$

Thus, for $\mathcal{L}^1-$a.e. sufficiently small $r > 0$, say, $0 < r < r_0 = r_0(x)$, (4.7.3) implies that

$$\frac{1}{2}\|\partial E\|(B(x,r)) \le \mathcal{H}^{n-1}(E\cap \partial B(x,r)). \tag{4.7.4}$$

This and (4.7.2) give

$$\|\partial(E\cap B(x,r))\|(\mathbb{R}^n) \le 3\mathcal{H}^{n-1}(E\cap \partial B(x,r)) \tag{4.7.5}$$

for $\mathcal{L}^1-$a.e. $0 < r < r_0$.
ii. Write $g(r) := \mathcal{L}^n(B(x,r)\cap E)$. Then by switching to polar coordinates,

$$g(r) = \int_0^r \mathcal{H}^{n-1}(\partial B(x,s)\cap E)\, ds,$$

so that g is absolutely continuous, and

$$g'(r) = \mathcal{H}^{n-1}(\partial B(x,r)\cap E)$$

for $\mathcal{L}^1-$a.e. $r > 0$. Applying now the isoperimetric inequality and (4.7.5), we calculate

$$\begin{aligned} g(r)^{1-\frac{1}{n}} &= \mathcal{L}^n(B(x,r)\cap E)^{1-\frac{1}{n}} \le C\|\partial(B(x,r)\cap E)\|(\mathbb{R}^n) \\ &\le C\mathcal{H}^{n-1}(E\cap \partial B(x,r)) \\ &= C_1 g'(r) \end{aligned}$$

for $\mathcal{L}^1-$a.e. $r \in (0, r_0)$. Thus

$$\frac{1}{C_1} \le g(r)^{\frac{1}{n}-1} g'(r) = n\left(g(r)^{\frac{1}{n}}\right)',$$

and so

$$g(r)^{\frac{1}{n}} \ge \frac{r}{C_1 n},$$

and

$$g(r) \ge \frac{r^n}{(C_1 n)^n}$$

for $0 < r < r_0$. This proves (i).

iii. Since for all $\phi \in \mathcal{C}_c^1(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$\int_E \operatorname{div}\phi \, d\mathcal{L}^n + \int_{\mathbb{R}^n\setminus E} \operatorname{div}\phi \, d\mathcal{L}^n = \int_{\mathbb{R}^n} \operatorname{div}\phi \, d\mathcal{L}^n = 0,$$

it is easy to verify that

$$\|\partial E\| = \|\partial(\mathbb{R}^n \setminus E)\|, \quad \nu_E = -\nu_{\mathbb{R}^n\setminus E}.$$

Consequently, (ii) follows from (i).

iv. According to the relative isoperimetric inequality,

$$\frac{\|\partial E\|(B(x,r))}{r^{n-1}} \geq C \min\left\{\frac{\mathcal{L}^n(B(x,r)\cap E)}{r^n}, \frac{\mathcal{L}^n(B(x,r)\setminus E)}{r^n}\right\}^{\frac{n-1}{n}},$$

and thus (iii) follows from (i) and (ii).

v. By (4.7.4),

$$\|\partial E\|(B(x,r)) \leq 2\mathcal{H}^{n-1}(E\cap \partial B(x,r)) \leq C r^{n-1}, \quad 0<r<r_0,$$

and this proves (iv).

vi. Assertion (v) follows from (4.7.2) and (iv). The proof is complete. □

. . . .

4.7.2. *Blowup.*

Definition (Hyperplane). *For each $x \in \partial^* E$, we define the* **hyperplane**

$$H(x) := \{y \in \mathbb{R}^n : \nu_E(x)\cdot(y-x) = 0\}.$$

Definition (Half–Space). *For each $x \in \partial^* E$, we define also the* **half–spaces**

$$H^+(x) := \{y \in \mathbb{R}^n : \nu_E(x)\cdot(y-x) \geq 0, \}$$
$$H^-(x) := \{y \in \mathbb{R}^n : \nu_E(x)\cdot(y-x) \leq 0\}.$$

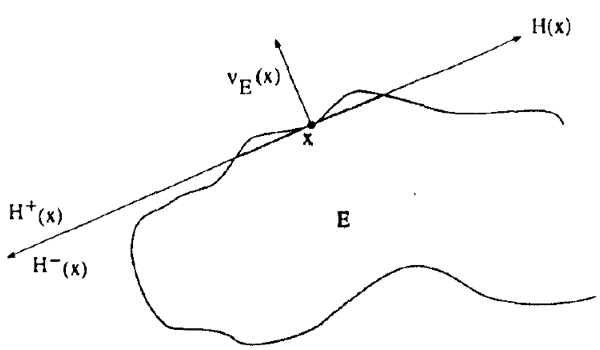

FIGURE 4.7.2. Half–Spaces.

We interpret $H(x)$ as an "approximate tangent plane" to $x \in \partial^* E$, in the sense that $H(x)$ is the set of all points y in $\mathbb{R}^n$ such that $y-x$ is orthogonal to the outer unit normal to E at x. Note that clearly $x \in H(x)$.

Remark. *Observe that* $y \in E \cap B(x,r)$ *if and only if* $g_r(y) \in E_r \cap B(x,1)$, *where* $g_r(y) := \frac{y-x}{r} + x$ *and* $E_r := \{y \in \Omega : |g_r(y)| > r\}$.

t5.7-1 **Theorem 4.7.1** (Blowup of Reduced Boundary). *Assume that* $x \in \partial^* E$. *Then*

$$\mathbb{1}_{E_r} \to \mathbb{1}_{H^-(x)} \quad \textit{in } L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

as $r \to 0$.

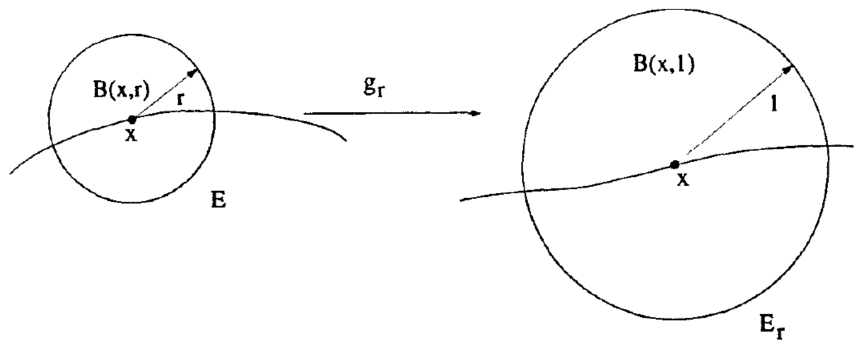

FIGURE 4.7.3. Blowup of Reduced Boundary.

Thus for small enough $r > 0$, $E \cap B(x,r)$ approximately equals the half ball $H^-(x) \cap B(x,r)$.

Proof.

i. Upon reorienting the boundary and axes if necessary, we may as well assume that

$$\begin{cases} x = 0,\ \nu_E(0) = e_n = (0,\dots,0,1), \\ H(0) = \{y \in \mathbb{R}^n : y_n = 0\}, \\ H^+(0) = \{y \in \mathbb{R}^n : y_n \geq 0\}, \\ H^-(0) = \{y \in \mathbb{R}^n : y_n \leq 0\}. \end{cases}$$

ii. Choose any sequence $r_k \to 0$. It suffices to show that there exists a subsequence $\{s_j\}_{j=1}^{+\infty} \subset \{r_k\}_{k=1}^{+\infty}$ such that

$$\mathbb{1}_{E_{s_j}} \to \mathbb{1}_{H^-(0)} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

iii. Fix $L > 0$ and put

$$D_r := E_r \cap B(0,L), \quad g_r(y) = \frac{y}{r}.$$

Then for any $\phi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$, $|\phi| \leq 1$, we have

$$\begin{aligned} \int_{D_r} \operatorname{div} \phi(z)\, d\mathcal{L}^n(z) &= \frac{1}{r^{n-1}} \int_{E \cap B(0,rL)} \operatorname{div}(\phi \circ g_r)(y)\, d\mathcal{L}^n(y) \\ &= \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} (\phi \cdot g_r) \cdot \nu_{E \cap B(0,rL)}\, d\|\partial(E \cap B(0,rL)) \\ &\leq \frac{\|\partial(E \cap B(0,rL))\|(\mathbb{R}^n)}{r^{n-1}} \\ &\leq C < +\infty \end{aligned}$$

for all $r > 0$ such that $0 < r < r_0$, for some $r_0 > 0$, according to Lemma (4.7.2). Consequently

$$\|\partial D_r\|(\mathbb{R}^n) \leq C < +\infty, \quad 0 < r < r_0,$$

and furthermore,

$$\|\mathbb{1}_{D_r}\|_{L^1(\mathbb{R}^n)} = \mathcal{L}^n(D_r) \leq \mathcal{L}^n(B(0,L)) < +\infty, \quad r > 0.$$

Hence

$$\|\mathbb{1}_{D_r}\|_{BV(\mathbb{R}^n)} \leq C < +\infty$$

for all $0 < r < r_0$.

In view of this estimate and Theorem (4.2.4), there exists a subsequence $\{s_j\}_{j=1}^{+\infty}$ of $\{r_k\}_{k=1}^{+\infty}$ and a function $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ such that, writing $E_j = E_{s_j}$, we have

$$\mathbb{1}_{E_j} \to f \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Passing to a further subsequence if necessary, we may also assume that $\mathbb{1}_{E_j} \to f$ $\mathcal{L}^n-$a.e. Hence, being the pointwise a.e. limit of indicator functions, $f(x) \in \{0,1\}$ for $\mathcal{L}^n-$a.e. x and so

$$f = \mathbb{1}_F \quad \mathcal{L}^n - \text{a.e.},$$

where $F \subset \mathbb{R}^n$ has locally finite perimeter. Hence if $\phi \in \mathcal{C}^1_c(\mathbb{R}^n;\mathbb{R}^n)$,

$$\int_F \operatorname{div}\phi \, d\mathcal{L}^n = \int_{\mathbb{R}^n} \phi \cdot \nu_F \, d\|\partial F\|, \tag{4.7.6}$$

for some $\|\partial F\|-$measurable function ν_F, with $|\nu_F| = 1$ $\|\partial F\|-$a.e.

To complete the proof, it remains to show that $F = H^-(0)$.

iv. We first claim that $\nu_F = e_n$ $\|\partial F\|-$a.e.

To show this, write $\nu_j := \nu_{E_j}$. Then if $\phi \in \mathcal{C}^1_c(\mathbb{R}^n;\mathbb{R}^n)$, it follows

$$\int_{\mathbb{R}^n} \phi \cdot \nu_j \, d\|\partial E_j\| = \int_{E_j} \operatorname{div}\phi \, d\mathcal{L}^n, \quad j \in \mathbb{N}.$$

Since

$$\mathbb{1}_{E_j} \to \mathbb{1}_F \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

we see from the above and (4.7.6) that

$$\int_{\mathbb{R}^n} \phi \cdot \nu_j \, d\|\partial E_j\| \to \int_{\mathbb{R}^n} \phi \cdot \nu_F \, d\|\partial F\| \quad \text{as } j \to +\infty.$$

Thus

$$\nu_j \|\partial E_j\| \rightharpoonup \nu_F \|\partial F\|$$

weakly in the sense of Radon measures. Consequently, for every $L > 0$ such that $\|\partial F\|(\partial B(0,L)) = 0$, and hence for all but at most countably many $L > 0$,

$$\int_{B(0,L)} \nu_j \, d\|\partial E_j\| \to \int_{B(0,L)} \nu_F \, d\|\partial F\|. \tag{4.7.7}$$

On the other hand, for all $\phi \in \mathcal{C}^1_c(\mathbb{R}^n;\mathbb{R}^n)$ as above,

$$\int_{\mathbb{R}^n} \phi \cdot \nu_j \, d\|\partial E_j\| = \frac{1}{s_j^{n-1}} \int_{\mathbb{R}^n} (\phi \circ g_{s_j}) \cdot \nu_E \, d\|\partial E\|,$$

so that

$$\begin{cases} \|\partial E_j\|(U(0,L)) = \frac{1}{s_j^{n-1}}\|\partial E\|(B(0,s_jL)), \\ \displaystyle\int_{B(0,L)} \nu_j \, d\|\partial E_j\| = \frac{1}{s_j^{n-1}} \int_{B(0,s_jL)} \nu_E \, d\|\partial E\|. \end{cases} \tag{4.7.8}$$

{eq:5.7-8

Therefore

$$\lim_{j\to+\infty} ⨍_{B(0,L)} \nu_j \, d\|\partial E_j\| = \lim_{j\to+\infty} ⨍_{B(0,s_jL)} \nu_E \, d\|\partial E\| = \nu_E(0) = e_n,$$

since $0 \in \partial^* E$. If $\|\partial F\|(\partial B(0,L)) = 0$, the Lower Semicontinuity Theorem (cf. Theorem (4.2.1)) implies that

$$\begin{aligned} \|\partial F\|(B(0,L)) &\le \liminf_{j\to+\infty} \|\partial E_j\|(B(0,L)) \\ &= \lim_{j\to+\infty} \int_{B(0,L)} e_n \cdot \nu_j \, d\|\partial E_j\| \\ &= \int_{B(0,L)} e_n \cdot \nu_F \, d\|\partial F\|, \end{aligned}$$

where the RHS follows from (4.7.7). Since $|\nu_F| = 1$ $\|\partial F\|-$a.e., the above inequality forces

$$\nu_F = e_n \quad \|\partial F\| - \text{a.e.}$$

It also follows from the above inequality that

$$\|\partial F\|(B(0,L)) = \lim_{j\to+\infty} \|\partial E_j\|(B(0,L))$$

whenever $\|\partial F\|(\partial B(0,L)) = 0$. This shows (iv).

v. We next show that F is a half–space.

To see this, by (iv), for all $\phi \in \mathcal{C}_c^1(\mathbb{R}^n;\mathbb{R}^n)$,

$$\int_F \operatorname{div}\phi \, d\mathcal{L}^n(z) = \int_{\mathbb{R}^n} \phi \cdot e_n \, d\|\partial F\|.$$

Fix $\epsilon > 0$ and let $f_\epsilon := \eta_\epsilon * \mathbb{1}_F$, where η_ϵ is the usual mollifier. Then $f_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$, and thus

$$\int_{\mathbb{R}^n} f_\epsilon \operatorname{div}\phi \, d\mathcal{L}^n(z) = \int_F \operatorname{div}(\eta_\epsilon * \phi) \, d\mathcal{L}^n(z) = \int_{\mathbb{R}^n} \eta_\epsilon * (\phi \cdot e_n) \, d\|\partial F\|.$$

On the other hand, integration by parts gives

$$\int_{\mathbb{R}^n} f_\epsilon \operatorname{div}\phi \, d\mathcal{L}^n(z) = -\int_{\mathbb{R}^n} Df_\epsilon \cdot \phi \, d\mathcal{L}^n(z).$$

Hence

$$\frac{\partial f_\epsilon}{\partial z_i} = 0, \quad i = 1, \dots, n-1, \quad \frac{\partial f_\epsilon}{\partial z_n} \le 0.$$

Since $f_\epsilon \to \mathbb{1}_F$ $\mathcal{L}^n-$a.e. as $\epsilon \to 0$, we conclude that, up to a set of $\mathcal{L}^n-$measure zero, for some $\gamma \in \mathbb{R}$,

$$F = \{y \in \mathbb{R}^n : y_n \le \gamma\}$$

vi. We lastly show that $F = H^-(0)$.

We must show that $\gamma = 0$ above. By contradiction, suppose instead that $\gamma > 0$. Since $\mathbb{1}_{E_j} \to \mathbb{1}_F$ in $L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\alpha(n)\gamma^n = \mathcal{L}^n(B(0,\gamma)\cap F) = \lim_{j\to+\infty} \mathcal{L}^n(B(0,\gamma)\cap E_j)$$

$$= \lim_{j\to+\infty} \frac{\mathcal{L}^n(B(0,\gamma s_j)\cap E)}{s_j^n},$$

a contradiction to Lemma (4.7.2)(ii).

Similarly, the case $\gamma < 0$ leads to a contradiction to Lemma (4.7.2)(i). □

We provide more detailed information regarding the blowup of E around a point $x \in \partial^* E$.

c5.7-1 **Corollary 4.7.1.** *Assume that $x \in \partial^* E$. Then*

(i) $\lim_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap E\cap H^+(x))}{r^n} = 0$ *and* $\lim_{r\to 0} \frac{\mathcal{L}^n((B(x,r)\setminus E)\cap H^-(x))}{r^n} = 0$, *and*

(ii) $\lim_{r\to 0} \frac{\|\partial E\|(B(x,r))}{\alpha(n-1)r^{n-1}} = 1$.

Definition (Measure Theoretic Unit Outer Normal)**.** *A unit vector $\nu_E(x)$ for which (i) holds with $H^\pm$ as defined above is called the* **measure theoretic unit outer normal** *to E at x.*

Proof.

i. We have

$$\begin{aligned}\frac{\mathcal{L}^n(B(x,r)\cap E\cap H^+(x))}{r^n} &= \mathcal{L}^n(B(x,1)\cap E_r\cap H^+(x))\\ &\to \mathcal{L}^n(B(x,1)\cap H^-(x)\cap H^+(x))\\ &= 0\end{aligned}$$

as $r \to 0$. The other limit in (i) is similar.

ii. Assume that $x = 0$. By (4.7.8),

$$\frac{\|\partial E\|(B(0,r))}{r^{n-1}} = \|\partial E_r\|(B(0,1)).$$

Since $\|\partial H^-(0)\|(\partial B(0,1)) = \mathcal{H}^{n-1}(\partial B(0,1)\cap H(0)) = 0$, part (ii) of the proof of Theorem (4.7.1) implies that

$$\begin{aligned}\lim_{r\to 0}\frac{\|\partial E\|(B(0,r))}{r^{n-1}} &= \|\partial H^-(0)\|(B(0,1))\\ &= \mathcal{H}^{n-1}(B(0,1)\cap H(0))\\ &= \alpha(n-1),\end{aligned}$$

as required. The proof is complete. □

. . . .

4.7.3. *Structure Theorem for Sets of Finite Perimeter.*

l5.7-3 **Lemma 4.7.3.** *There exists a constant $C > 0$, depending only on n, such that*

$$\mathcal{H}^{n-1}(B) \le C\|\partial E\|(B)$$

for all $B \subset \partial^*(E)$.

Proof. Let $\epsilon, \delta > 0$, and $B \subset \partial^* E$. Since $\|\partial E\|$ is a Radon measure, there exists an open set U containing B such that

$$\|\partial E\|(U) \leq \|\partial E\|(B) + \epsilon.$$

Now according to Lemma (4.7.2), if $x \in \partial^* E$, then

$$\liminf_{r \to 0} \frac{\|\partial E\|(B(x,r))}{r^{n-1}} > A_3 > 0.$$

Put

$$\mathcal{F} := \left\{ B(x,r) : x \in B, B(x,r) \subset U, r < \frac{\delta}{10}, \|\partial E\|(B(x,r)) > A_3 r^{n-1} \right\}.$$

By the Vitali $5r-$Covering Lemma, there exist disjoint balls $\{B(x_i,r_i)\}_{i=1}^{+\infty} \subset \mathcal{F}$ such that

$$B \subset \bigcup_{i=1}^{+\infty} B(x_i, 5r_i).$$

Since $r_i < \frac{\delta}{10}$, so that $\operatorname{diam}(B(x_i,5r_i)) \leq \delta$ for each $i \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{H}^{n-1}_\delta(B) &\leq \sum_{i=1}^{+\infty} \alpha(n-1)(5r_i)^{n-1} \leq C \sum_{i=1}^{+\infty} \|\partial E\|(B(x_i,r_i)) \\ &\leq C\|\partial E\|(U) \\ &\leq C(\|\partial E\|(B) + \epsilon). \end{aligned}$$

Letting $\epsilon \to 0$ and then $\delta \to 0$, we find

$$\mathcal{H}^{n-1}(B) \leq C\|\partial E\|(B),$$

as required. The proof is complete. □

Finally, we show that a set of locally finite perimeter has a "measure theoretically $\mathcal{C}^1$ boundary."

t5.7-2 **Theorem 4.7.2** (Structure Theorem for Sets of Finite Perimeter)**.** *Assume that E has locally finite perimeter in $\mathbb{R}^n$. Then*

(i)

$$\partial^* E = \left(\bigcup_{k=1}^{+\infty} K_k \right) \cup N,$$

where

$$\|\partial E\|(N) = 0$$

and K_k is a compact subset of a $\mathcal{C}^1$ hypersurface S_k, $k \in \mathbb{N}$;

(ii) Furthermore, $\nu_E|_{S_k}$ is normal to S_k, $k \in \mathbb{N}$;

(iii) $\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial^* E$.

Remark. *Theorem (4.7.2) states that the reduced boundary $\partial^* E$ is a countable union of compact subsets of $\mathcal{C}^1-$smooth hypersurfaces with a set of $\|\partial E\|-$measure zero, on which an outward unit normal is defined and given by ν_E, and lastly that the perimeter measure $\|\partial E\|$ of E is just $(n-1)-$dimensional Hausdorff measure restricted to the reduced boundary $\partial^* E$.*

Proof.

i. For each $x \in \partial^* E$, we have according to Corollary (4.7.1) that

$$\begin{cases} \lim\limits_{r\to 0} \dfrac{\mathcal{L}^n(B(x,r)\cap E\cap H^+(x))}{r^n} = 0, \\ \lim\limits_{r\to 0} \dfrac{\mathcal{L}^n((B(x,r)\setminus E)\cap H^-(x))}{r^n} = 0. \end{cases} \tag{4.7.9}$$

Using Egorov's Theorem, we obtain disjoint $\|\partial E\|-$measurable sets $\{F_i\}_{i=1}^{+\infty} \subset \partial^* E$ such that

$$\begin{cases} \|\partial E\| \left(\partial^* E \setminus \bigcup_{i=1}^{+\infty} F_i\right) = 0, \quad \|\partial E\|(F_i) < +\infty, \\ \text{the convergence in (4.7.9) is uniform for } x \in F_i,\ i \in \mathbb{N}. \end{cases}$$

Then by Lusin's Theorem, for each $i \in \mathbb{N}$ there exist disjoint compact sets $\{E_i^j\}_{j=1}^{+\infty} \subset F_i$ such that

$$\begin{cases} \|\partial E\| \left(F_i \setminus \bigcup_{j=1}^{+\infty} E_i^j\right) = 0, \\ \nu_E|_{E_i^j} \quad \text{is continuous.} \end{cases}$$

Reindex the sets $\{E_i^j\}_{j=1}^{+\infty}$ and call the collection $\{K_k\}_{k=1}^{+\infty}$. Then

$$\begin{cases} \partial^* E = \left(\bigcup_{k=1}^{+\infty} K_k\right) \cup N, \quad \|\partial E\|(N) = 0, \\ \text{the convergence in (4.7.9) is uniform on } K_k, \\ \nu_E|_{K_k} \quad \text{is continuous},\ k \in \mathbb{N}. \end{cases} \tag{4.7.10}$$

ii. Define for any $\delta > 0$

$$\rho_k(\delta) := \sup\left\{ \frac{|\nu_E(x)\cdot(y-x)|}{|y-x|} : 0 < |x-y| \le \delta,\ x,y \in K_k \right\}.$$

Note that $\rho_k(\delta)$ may be interpreted as a sort of upper Lipschitz constant of the inner product of $\nu_E(x)\cdot(y-x)$ over all $x,y \in K_k$ such that $\operatorname{dist}(x,y)$ is small enough.

iii. We next claim that for each $k \in \mathbb{N}$, $\rho_k(\delta) \to 0$ as $\delta \to 0$.

To see this, we first may as well assume that $k = 1$. Fix $0 < \epsilon < 1$. By (4.7.9) and (4.7.10) there exists $0 < \delta < 1$ such that if $z \in K_1$ and $r < 2\delta$, then

$$\begin{cases} \mathcal{L}^n(E\cap B(z,r)\cap H^+(z)) < \dfrac{\epsilon^n}{2^{n+2}}\alpha(n) r^n, \\ \mathcal{L}^n(E\cap B(z,r)\cap H^-(z)) > \alpha(n)\left(\dfrac{1}{2} - \dfrac{\epsilon^n}{2^{n+2}}\right) r^n. \end{cases} \tag{4.7.11}$$

Assume now that $x, y \in K_1$ with $0 < |x-y| \le \delta$.

Case 1: $\nu_E(x)\cdot(y-x) > \epsilon|x-y|$.

In this case, since $\epsilon < 1$, we have

$$B(y,\epsilon|x-y|) \subset H^+(x)\cap B(x,2|x-y|). \tag{4.7.12}$$

To see this, observe that if $z \in B(y,\epsilon|x-y|)$, then $z = y + w$, where $|w| \le \epsilon|x-y|$, so that

$$\nu_E(x)\cdot(z-x) = \nu_E(x)\cdot(y-x) + \nu_E(x)\cdot w > \epsilon|x-y| - |w| \ge 0.$$

On the other hand, (4.7.11) with $z = x$ implies that

$$\mathcal{L}^n(E\cap B(x,2|x-y|)\cap H^+(x)) < \frac{\epsilon^n}{2^{n+2}}\alpha(n)(2|x-y|)^n$$

$$= \frac{\epsilon^n \alpha(n)}{4}|x-y|^n,$$

and (4.7.11) with $z = y$ implies that

$$\begin{aligned}\mathcal{L}^n(E \cap B(y, \epsilon|x-y|)) &\geq \mathcal{L}^n(E \cap B(y, \epsilon|x-y|) \cap H^-(y)) \\ &\geq \frac{\epsilon^n \alpha(n)|x-y|^n}{2}\left(1 - \frac{\epsilon^n}{2^{n+1}}\right) \\ &> \frac{\epsilon^n \alpha(n)}{4}|x-y|^n.\end{aligned}$$

However, applying $\mathcal{L}^n \llcorner E$ to both sides of (4.7.12) yields the estimate

$$\mathcal{L}^n(E \cap B(y, \epsilon|x-y|)) \leq \mathcal{L}^n(E \cap B(x, 2|x-y|) \cap H^+(x)),$$

which is impossible according to the above inequalities.

Case 2: $\nu_E(x) \cdot (y-x) < -\epsilon|x-y|$.

This case similarly leads to a contradiction.

Thus it must be the case that $|\nu_E(x) \cdot (y-x)| \leq \epsilon|y-x|$ for all $x, y \in K_1$ with $0 < |x-y| < \delta$. Hence $\rho_k(\delta) \to 0$ as $\delta \to 0$ for each $k \in \mathbb{N}$, as required. This proves the claim in (iii).

iv. We now apply the Whitney Extension theorem with

$$f = 0, \quad d = \nu_E \text{ on } K_k.$$

We conclude that there exist $\mathcal{C}^1$−functions $\overline{f}_k : \mathbb{R}^n \to \mathbb{R}$ such that

$$\overline{f}_k = 0 \quad \text{on } K_k, D\overline{f}_k = \nu_E \quad \text{on } K_k.$$

Put

$$S_k := \left\{x \in \mathbb{R}^n : \overline{f}_k = 0, \ |D\overline{f}_k| > \frac{1}{2}\right\}, \quad k \in \mathbb{N}.$$

By the implicit function theorem, S_k is a $\mathcal{C}^1$, $(n-1)$−dimensional submanifold of $\mathbb{R}^n$. Clearly $K_k \subset S_k$. This proves assertions (i) and (ii) of the theorem.

v. Choose a Borel set $B \subset \partial^* E$. According to Lemma (4.7.3),

$$\mathcal{H}^{n-1}(B \cap N) \leq C\|\partial E\|(B \cap N) = 0.$$

Thus we may as well assume that $B \subset \cup_{k=1}^{+\infty} K_k$, and in fact $B \subset K_1$. By assertion (ii), there exists a $\mathcal{C}^1$−hypersurface $S_1 \supset K_1$. Let

$$\nu := \mathcal{H}^{n-1} \llcorner S_1.$$

Since S_1 is $\mathcal{C}^1$,

$$\lim_{r \to 0} \frac{\nu(B(x,r))}{\alpha(n-1)r^{n-1}} = 1, \quad x \in B.$$

Thus Corollary (4.7.1)(ii) implies that

$$\lim_{r \to 0} \frac{\nu(B(x,r))}{\|\partial E\|(B(x,r))} = \frac{\lim_{r\to 0} \frac{\nu(B(x,r))}{\alpha(n-1)r^{n-1}}}{\lim_{r \to 0} \frac{\|\partial E\|(B(x,r))}{\alpha(n-1)r^{n-1}}} = 1, \quad x \in B.$$

Since ν and $\|\partial E\|$ are Radon measures, we obtain

$$\|\partial E\|(B) = \nu(B) = \mathcal{H}^{n-1}(B),$$

as required. Note that this is for any $B \subset \partial^* E$. The proof is complete. □

References

1. Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992. MR 1158660