CHANGE OF COORDINATES IN TRIPLE INTEGRATION

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1. Preliminaries

This last week in class, we discussed the computation of triple integrals in cylindrical and spherical coordinates. In particular, we found that cylindrical coordinates are particularly useful when the projection of our region of integration to the $xy-$plane can easily be described in polar coordinates (spheres, cones, and cylinders are all good examples), and spherical coordinates are nice when we have *radial symmetry* – symmetry about a point in 3–space (think spheres and cones).

In class, I briefly tried to demonstrate that changing coordinates in integration is just "jacked–up u–substitution." In these notes I will attempt to elaborate further on these similarities.

2. A Review of U–Substitution

Our discussion of cylindrical and spherical coordinates will be very abstract, so let us first consider the following example of u–substitution from single variable calculus to build some intuition.

Example 2.1. *Consider the problem of evaluating the integral*

$$\int_0^1 xe^{2x^2}\ dx.$$

From single–variable calculus, we know to let $u = 2x^2$*, and we compute* $du = 4x\ dx$*. Thus the integration problem becomes*

$$\int_0^2 e^u\ \frac{du}{4},$$

and evaluating the integral, we find that

$$\begin{aligned}\int_0^1 xe^{2x^2}\ dx &= \int_0^2 e^u \frac{du}{4}\\ &= \frac{e^u}{4}\bigg|_0^2\\ &= \frac{e^2}{4} - \frac{1}{4}.\end{aligned}$$

We will see that there is quite a bit of machinery which is obscured in this above process. In fact what we have done here is make the following coordinate transformation in order to make this integration easier to calculate:

$$\Psi : [0,1] \subseteq \mathbb{R} \to [0,2] \subseteq \mathbb{R},$$
$$\Psi : x \mapsto 2x^2 = u.$$

That is, the function Ψ takes in $x-$coordinates and spits out $u-$coordinates.

One other note: the function Ψ here is differentiable. Thus when we make this coordinate transformation, we are also inducing a transformation on the level of *derivatives*: this is precisely the reason we need to compute $du = 4x\ dx$. If we differentiate Ψ with respect to x, we compute

$$\Psi'(x) = 4x,$$

and we see that this matches exactly the $4x$ term in the $u-$substitution "$du = 4x\ dx$." As we will see in later sections, this factor $4x$ is called the *Jacobian determinant* of the transformation (sometimes this factor is also confusingly just called the *Jacobian* out of laziness, and as we will see, the Jacobian is a matrix of all the partial derivatives of a function – it is not a number).

Let us now move on to a discussion of cylindrical and spherical coordinates in $\mathbb{R}^3$, and try to compare triple integration in these coordinate systems with standard $u-$substitution from single variable calculus.

3. Cylindrical Coordinates

Let us fix a point $P(r, \theta, z) \in \mathbb{R}^3$ in cylindrical coordinates in our minds. We recall the following equations from our discussion of cylindrical coordinates:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{aligned} \tag{3.1}$$

We may write these equations (3.1) conveniently as the following parameterization from

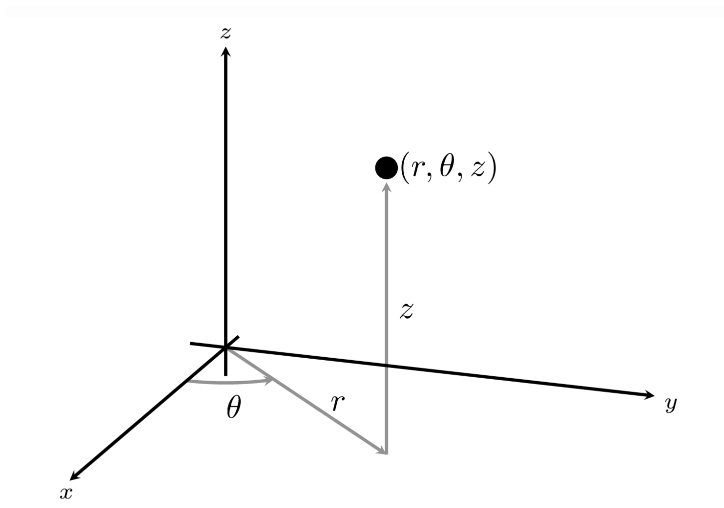

Figure 1. Cylindrical Coordinates.

cylindrical coordinates to rectangular coordinates:

$$\Psi : \mathbb{R}^3_{r,\theta,z} \to \mathbb{R}^3_{x,y,z},$$
$$\Psi : (r,\theta,z) \mapsto (r\cos\theta, r\sin\theta, z) = (x,y,z). \tag{3.2}$$

To reiterate: Ψ here is a vector–valued function which takes in cylindrical coordinates and spits out rectangular coordinates by the formula

$$\Psi(r,\theta,z) = (r\cos\theta, r\sin\theta, z).$$

Now let us consider the problem of integrating a function $f(x,y,z)$ over some cylindrical region E described by

$$E = \begin{cases} \alpha \le \theta \le \beta, \\ a \le r \le b, \\ c \le z \le d. \end{cases}$$

Using the equations (3.1) and our description of the region E, what we find is that

$$\iiint_{\Psi(E)} f(x,y,z)\ dV_{x,y,z} = \iiint_E f(r\cos\theta, r\sin\theta, z)\ dV_{r,\theta,z}.$$

The only matter left to discuss is how we convert $dV_{x,y,z}$ to $dV_{r,\theta,z}$, that is, how to convert something like $dxdydz$ to $dzdrd\theta$ (think about how we have to convert dx to du in $u-$substitution.) If our change in coordinates in $3-$space is to have any relation whatsoever with $u-$substitution in single–variable calculus, we should expect to get a "scaling factor" in this conversion, and we should expect this scaling factor to be given by "differentiating" $dxdydz$, similarly to how we "differentiated" du to get $4x\ dx$ in our Example (2.1).

The question we should ask ourselves here is how do we differentiate $dxdydz$? If we compare with Example (2.1), our coordinate transformation in our example was $\Psi(x) = 2x^2$, and, upon differentiating, we found that $\Psi'(x) = 4x$, which is exactly the scaling factor we wanted in our integration. Note by (3.2) that our coordinate transformation here is $\Psi(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$. But the question that still remains is how can we differentiate $\Psi(r,\theta,z)$ here? Well, Ψ is a function of three variables r, θ, and z, and each component function $(r\cos\theta, r\sin\theta, z)$ of $\Psi(r,\theta,z)$ is itself differentiable with respect to each of these variables. So, a reasonable way to think of differentiating Ψ would be to differentiate each component function with respect to each variable r, θ, and z. This is in fact how we will go about differentiating Ψ, but first let us define the following functions to make our notation (and thus, our future calculations) simpler:

$$\psi_1(r,\theta,z) = r\cos\theta,$$
$$\psi_2(r,\theta,z) = r\sin\theta,$$
$$\psi_3(r,\theta,z) = z.$$

Observe here that we may write

$$\begin{aligned}\Psi(r,\theta,z) &= (r\cos\theta, r\sin\theta, z) \\ &= (\psi_1(r,\theta,z), \psi_2(r,\theta,z), \psi_3(r,\theta,z)).\end{aligned}$$

Thus how we are going to differentiate $\Psi(r,\theta,z)$ is by differentiating each of $\psi_1(r,\theta,z), \psi_2(r,\theta,z)$, and $\psi_3(r,\theta,z)$ with respect to r, θ, and z. In fact, we will form the following matrix of partial derivatives:

$$D\Psi = \begin{bmatrix} \frac{\partial \psi_1}{\partial r} & \frac{\partial \psi_1}{\partial \theta} & \frac{\partial \psi_1}{\partial z} \\ \frac{\partial \psi_2}{\partial r} & \frac{\partial \psi_2}{\partial \theta} & \frac{\partial \psi_2}{\partial z} \\ \frac{\partial \psi_3}{\partial r} & \frac{\partial \psi_3}{\partial \theta} & \frac{\partial \psi_3}{\partial z} \end{bmatrix}.$$

Upon computing each of the partial derivatives in the above matrix $D\Psi$, what we should find is

$$D\Psi = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is called the *Jacobian matrix* of the transformation $\Psi(r, \theta, z)$.

Let us take a step back and recall what we are trying to accomplish in this process. We want to set up the integral

$$\iiint_E f(r\cos\theta, r\sin\theta, z)\ dV_{r,\theta,z}$$

in cylindrical coordinates. To do this, we need to convert $dxdydz$ to $dzdrd\theta$. Further, to do this, we need to include a scaling factor that tells us how our coordinate transformation $\Psi(r, \theta, z)$ affects volumes of cylindrical regions like E.

The problem we have right now is that our Jacobian matrix $D\Psi$ is just that – a matrix, not a number we can take as our scaling factor. So the question becomes: can we get a number out of $D\Psi$ that we can take as our scaling factor (one that tells us how the volume of regions like E change under our coordinate transformation Ψ)? The answer is *yes*.

Let us recall from linear algebra that the *determinant* of a 3×3 matrix tells us how the volume of a box changes under our transformation (if you haven't taken linear algebra yet, it is okay to take this as a black box for now). Specifically, since the partial derivatives with respect to r, θ, and z tell us how much a given function changes in infinitesimal intervals around r, θ, and z, respectively, we can view our Jacobian matrix $D\Psi$ as telling us how much the volume of an infinitesimal box around (r, θ, z) changes under our transformation $\Psi(r, \theta, z)$. Thus, if we want our triple integral $\iiint_E f(r\cos\theta, r\sin\theta, z)\ dV_{r,\theta,z}$ to tell us anything about the volume of our region E, the *determinant* of the Jacobian matrix $D\Psi$ is exactly the scaling factor we want to include in our integration – again, this is because the determinant tells us how our coordinate transformation affects volumes. Thus, what our integral will become is:

$$\iiint_E f(r\cos\theta, r\sin\theta, z)\ dV_{r,\theta,z} = \int_\alpha^\beta \int_a^b \int_c^d f(r\cos\theta, r\sin\theta, z) \cdot |\det(D\Psi)|\ dzdrd\theta. \quad (3.3)$$

We compute the following determinant by cofactor expansion (again, if you haven't taken linear algebra yet, think about how we computed cross products early on in the semester):

$$\begin{aligned} \det(D\Psi) &= \det\left(\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\ &= \cos\theta \det\left(\begin{bmatrix} r\cos\theta & 0 \\ 0 & 1 \end{bmatrix}\right) - (-r\sin\theta)\det\left(\begin{bmatrix} \sin\theta & 0 \\ 0 & 1 \end{bmatrix}\right) + 0\det\left(\begin{bmatrix} \sin\theta & r\cos\theta \\ 0 & 0 \end{bmatrix}\right) \end{aligned}$$

$$= (\cos\theta)(r\cos\theta) + (r\sin\theta)(\sin\theta)$$
$$= r(\cos^2\theta + \sin^2\theta)$$
$$= r.$$

Inserting this calculation in for $\det(D\Psi)$ in (3.3), we obtain

$$\iiint_E f(r\cos\theta, r\sin\theta, z)\ dV_{r,\theta,z} = \int_\alpha^\beta \int_a^b \int_c^d f(r\cos\theta, r\sin\theta, z)\ r dz dr d\theta,$$

which is exactly the formula we learned in class.

4. A Computation Using Differential Forms

A more advanced computation of this scaling factor r is as follows. The objects "dx," "dy," and "dz" that appear in triple integration have a name: they are called *1–forms,* or *covectors.* You can think of a 1–form as a function that takes in vectors and spits out numbers. For example, let us fix the vector $\mathbf{u} = \langle 1,2,3\rangle$ in our minds. The 1–form dx tells us the $x-$coordinate of the vector $\mathbf{u}$, dy tells us the $y-$coordinate of $\mathbf{u}$, and dz tells us the $z-$coordinate of u. That is,

$$dx(\mathbf{u}) = dx(\langle 1,2,3\rangle) = 1,$$
$$dy(\mathbf{u}) = dy(\langle 1,2,3\rangle) = 2,$$
$$dz(\mathbf{u}) = dz(\langle 1,2,3\rangle) = 3.$$

Now recall from Equations (3.1) that

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = z.$$

Since our integral in cylindrical coordinates involves dr, $d\theta$, and dz, and not dx, dy, and dz, we want to compute what $dr, d\theta$, and dz are using the above equations. If we differentiate both sides of the above equations, we obtain

$$\begin{aligned} dx &= d(r\cos\theta), \\ dy &= d(r\sin\theta), \\ dz &= d(z). \end{aligned} \tag{4.1}$$

Now we may treat "d" like the derivative, so it respects all the rules the usual derivative does which you have learned from single variable and multivariable calculus. We compute the right–hand sides of Equations (4.1):

$$\begin{aligned} dx &= d(r\cos\theta) = (dr)\cos\theta + r\cdot d(\cos\theta) = \cos\theta\ dr - r\sin\theta\ d\theta, \\ dy &= d(r\sin\theta) = (dr)\sin\theta + r\cdot d(\sin\theta) = \sin\theta\ dr + r\cos\theta\ d\theta, \\ dz &= dz, \end{aligned} \tag{4.2}$$

where the second equality in lines 1 and 2 follow from product rule, and the third equality in lines 1 and 2 follows from chain rule on $\cos\theta$ and $\sin\theta$, respectively. Compare these Equations (4.2) with the Jacobian $D\Psi$ found in the previous section.

Note that we have written the 1–forms dx, dy, and dz in terms of the 1–forms dr, $d\theta$, and dz. We may define a multiplication rule between two 1–forms to get a 2-*form,* and between three 1–forms to get a 3-*form.* The rule we need to know is that whenever we swap the order of two 1–forms, we need to negate the result. For instance:

$$dxdy = -dydx.$$

Note that this means that $drdr$, $d\theta d\theta$, and $dzdz$ are all zero, because:

$$drdr = -drdr \implies 2drdr = 0 \implies drdr = 0,$$

and similarly for $d\theta d\theta$ and $dzdz$. This multiplication rule is made to coincide with taking the determinant of a matrix (if we swap two rows of a matrix, we need to negate its determinant).

Since our cylindrical integral uses the order $dzdrd\theta$, we will express our 3–form in this order as well. Let us first compute $dxdy$. We find, using Equations (4.2):

$$\begin{aligned} dxdy &= (\cos\theta\ dr - r\sin\theta\ d\theta)(\sin\theta\ dr + r\cos\theta\ d\theta) \\ &= \cos\theta\sin\theta\ drdr + r\cos^2\theta\ drd\theta - r\sin^2\theta\ d\theta dr - r^2\sin\theta\cos\theta\ d\theta d\theta \\ &= r\cos^2\theta\ drd\theta - r\sin^2\theta\ d\theta dr \\ &= r\cos^2\theta\ drd\theta + r\sin^2\theta\ drd\theta \\ &= r(\cos^2\theta + \sin^2\theta)\ drd\theta \\ &= r\ drd\theta, \end{aligned}$$

where the third line follows from the fact that $drdr = 0$ and $d\theta d\theta = 0$, and the fourth line follows from swapping the order of $d\theta dr$ with $drd\theta$. That is,

$$dxdy = r\ drd\theta,$$

which should be familiar from polar coordinates. If we multiply on the right by dz, we find that

$$dxdydz = r\ drd\theta dz = -r\ drdzd\theta = r\ dzdrd\theta,$$

and we see again that this scaling factor r falls out naturally.

As a last side note to close this section, we call $dxdydz$ the *volume form* of $\mathbb{R}^3$ in rectangular coordinates, and $r\ dzdrd\theta$ is called the *volume form* of $\mathbb{R}^3$ in cylindrical coordinates.

5. Spherical Coordinates

We now turn to a discussion of spherical coordinates, so let us fix a new point $P(\rho, \theta, \phi) \in \mathbb{R}^3$ in our minds. Recall the following equations from class:

$$\begin{aligned} x &= \rho\sin\phi\cos\theta, \\ y &= \rho\sin\phi\sin\theta, \\ z &= \rho\cos\phi. \end{aligned} \tag{5.1}$$

We may write these Equations (5.1) as the following coordinate transformation:

$$\begin{aligned} \Psi &: \mathbb{R}^3_{\rho,\theta,\phi} \to \mathbb{R}^3_{x,y,z}, \\ \Psi &: (\rho, \theta, \phi) \mapsto (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) = (x, y, z). \end{aligned} \tag{5.2}$$

Here Ψ is a vector–valued function that takes in spherical coordinates and spits out rectangular coordinates by the rule

$$\Psi(\rho, \theta, \phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

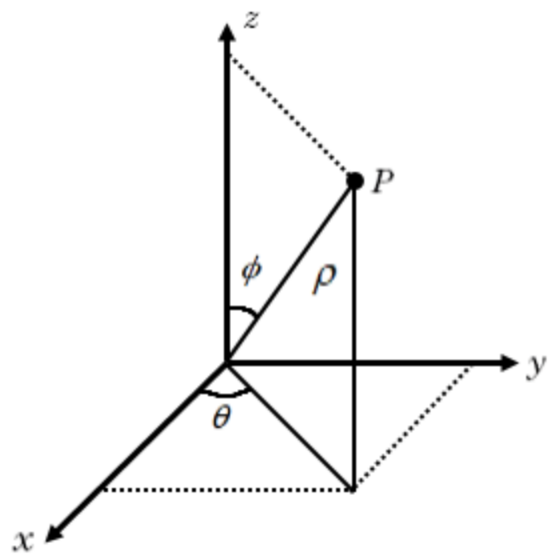

FIGURE 2. Spherical Coordinates.

We again want to consider integrating some function $f(x, y, z)$ over a new spherical region E :

$$E = \begin{cases} \alpha \leq \theta \leq \beta, \\ \gamma \leq \phi \leq \delta, \\ a \leq \rho \leq b. \end{cases}$$

Combining Equations (5.1) and the bounds given by our region E, we see that we want to compute the following integral:

$$\iiint_{\Psi(E)} f(x, y, z)\ dV_{x,y,z} = \iiint_E f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\ dV_{\rho,\theta,\phi}.$$

We see again that we have to convert $dV_{x,y,z}$ to $dV_{\rho,\theta,\phi}$. We will again compute the Jacobian determinant of the transformation. Let us write

$$\begin{aligned} \psi_1(\rho, \theta, \phi) &= \rho \sin\phi \cos\theta, \\ \psi_2(\rho, \theta, \phi) &= \rho \sin\phi \sin\theta, \\ \psi_3(\rho, \theta, \phi) &= \rho \cos\phi, \end{aligned}$$

so that we have

$$\begin{aligned} \Psi(\rho, \theta, \phi) &= (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \\ &= (\psi_1(\rho, \theta, \phi), \psi_2(\rho, \theta, \phi), \psi_3(\rho, \theta, \phi)). \end{aligned}$$

Thus our Jacobian matrix $D\Psi$ can be written as

$$D\Psi = \begin{bmatrix} \dfrac{\partial\psi_1}{\partial\rho} & \dfrac{\partial\psi_1}{\partial\theta} & \dfrac{\partial\psi_1}{\partial\phi} \\ \dfrac{\partial\psi_2}{\partial\rho} & \dfrac{\partial\psi_2}{\partial\theta} & \dfrac{\partial\psi_2}{\partial\phi} \\ \dfrac{\partial\psi_3}{\partial\rho} & \dfrac{\partial\psi_3}{\partial\theta} & \dfrac{\partial\psi_3}{\partial\phi} \end{bmatrix},$$

and, after computing each of the partial derivatives in $D\Psi$, we should have

$$D\Psi = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}.$$

Since the determinant of this matrix tells us how volume changes under the transformation Ψ, we must again include this information in our spherical integral. Hence, the integral we want is:

$$\begin{aligned} &\iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ dV_{\rho,\theta,\phi} \\ &\qquad = \int_\alpha^\beta \int_\gamma^\delta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \cdot |\det(D\Psi)|\ d\rho d\phi d\theta. \qquad (5.3) \end{aligned}$$

Computing this determinant $\det(D\Psi)$ by cofactor expansion, we obtain

$$\begin{aligned} \det(D\Psi) &= \det\left(\begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}\right) \\ &= \sin\phi\cos\theta\cdot\det\left(\begin{bmatrix} \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ 0 & -\rho\sin\phi \end{bmatrix}\right) - \\ &\qquad (-\rho\sin\phi\sin\theta)\cdot\det\left(\begin{bmatrix} \sin\phi\sin\theta & \rho\cos\phi\sin\theta \\ \cos\phi & -\rho\sin\phi \end{bmatrix}\right) + \\ &\qquad \rho\cos\phi\cos\theta\cdot\det\left(\begin{bmatrix} \sin\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & 0 \end{bmatrix}\right) \\ &= (\sin\phi\cos\theta)\cdot(-\rho^2\sin^2\phi\cos\theta) + (\rho\sin\phi\sin\theta)\cdot(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta) + \\ &\qquad (\rho\cos\phi\cos\theta)\cdot(-\rho\sin\phi\cos\theta\cos\phi) \\ &= -\rho^2\sin^3\phi\cos^2\theta - \rho^2\sin^3\phi\sin^2\theta - \rho^2\sin\phi\sin^2\theta\cos^2\phi - \rho^2\sin\phi\cos^2\phi\cos^2\theta \\ &= (-\rho^2\sin^3\phi)\cdot(\cos^2\theta + \sin^2\theta) - (\rho^2\sin\phi\cos^2\phi)\cdot(\sin^2\theta + \cos^2\theta) \\ &= -\rho^2\sin^3\phi - \rho^2\sin\phi\cos^2\phi \\ &= (-\rho^2\sin\phi)\cdot(\sin^2\phi + \cos^2\phi) \\ &= -\rho^2\sin\phi. \end{aligned}$$

Noting that we restrict ϕ to lie in the interval $0 \le \phi \le \pi$, where $\sin\phi \ge 0$, taking the absolute value $|\det(D\Psi)|$ gives

$$|\det(D\Psi)| = \rho^2\sin\phi,$$

as we know we wanted from class. Inserting this factor into the integral in (5.3), we obtain

$$\begin{aligned}&\int_\alpha^\beta\int_\gamma^\delta\int_a^b f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\cdot|\det(D\Psi)|\;d\rho d\phi d\theta\\&\qquad=\int_\alpha^\beta\int_\gamma^\delta\int_a^b f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\cdot\rho^2\sin\phi\;d\rho d\phi d\theta,\end{aligned}$$

which is the spherical integral we found in class.

The analogous computation using differential forms gets very messy. Recalling from Equations (5.1) that

$$\begin{aligned}x&=\rho\sin\phi\cos\theta,\\y&=\rho\sin\phi\sin\theta,\\z&=\rho\cos\phi.\end{aligned}$$

We compute, using the product rule,

$$\begin{aligned}dx&=\sin\phi\cos\theta\;d\rho+\rho\cos\phi\cos\theta\;d\phi-\rho\sin\phi\sin\theta\;d\theta,\\dy&=\sin\phi\sin\theta\;d\rho+\rho\cos\phi\sin\theta\;d\phi+\rho\sin\phi\cos\theta\;d\theta,\\dz&=\cos\phi\;d\rho-\rho\sin\phi\;d\phi\end{aligned}\tag{5.4}$$

(compare again these equations with the Jacobian $D\Psi$). We first multiply dx and dy, to find

$$\begin{aligned}dxdy&=(\sin\phi\cos\theta\;d\rho+\rho\cos\phi\cos\theta\;d\phi-\rho\sin\phi\sin\theta\;d\theta)\times\\&\qquad(\sin\phi\sin\theta\;d\rho+\rho\cos\phi\sin\theta\;d\phi+\rho\sin\phi\cos\theta\;d\theta)\\&=\sin^2\phi\cos\theta\sin\theta\;d\rho d\rho+\rho\sin\phi\cos\theta\cos\phi\sin\theta\;d\rho d\theta+\\&\qquad\rho\sin^2\phi\cos^2\theta\;d\rho d\theta+\rho\cos\phi\cos\theta\sin\phi\sin\theta\;d\phi d\rho+\\&\qquad\rho^2\cos^2\phi\cos\theta\sin\theta\;d\phi d\phi+\rho^2\cos\phi\cos^2\theta\sin\phi\;d\phi d\theta-\\&\qquad\rho\sin^2\phi\sin^2\theta\;d\theta d\rho-\rho^2\sin\phi\sin^2\theta\cos\phi\;d\theta d\phi-\\&\qquad\rho^2\sin^2\phi\sin\theta\cos\theta\;d\theta d\theta\\&=\rho\sin\phi\cos\theta\cos\phi\sin\theta+\rho\sin^2\phi\cos^2\theta\;d\rho d\theta+\\&\qquad\rho\cos\phi\cos\theta\sin\phi\sin\theta\;d\phi d\rho+\rho^2\cos\phi\cos^2\theta\sin\phi\;d\phi d\theta-\\&\qquad\rho\sin^2\phi\sin^2\theta\;d\theta d\rho-\rho^2\sin\phi\sin^2\theta\cos\phi\;d\theta d\phi\\&=\rho\sin\phi\cos\theta\cos\phi\sin\theta\;d\rho d\phi-\rho\cos\phi\cos\theta\sin\phi\sin\theta\;d\rho d\phi+\\&\qquad\rho\sin^2\phi\cos^2\theta\;d\rho d\theta+\rho\sin^2\phi\sin^2\theta\;d\rho d\theta+\\&\qquad\rho^2\cos\phi\cos^2\theta\sin\phi\;d\phi d\theta+\rho^2\sin\phi\sin^2\theta\cos\phi\;d\phi d\theta\\&=\rho\sin^2\phi(\cos^2\theta+\sin^2\theta)\;d\rho d\theta+\rho^2\sin\phi\cos\phi(\cos^2\theta+\sin^2\theta)\;d\phi d\theta\\&=\rho\sin^2\phi\;d\rho d\theta+\rho^2\sin\phi\cos\phi\;d\phi d\theta,\end{aligned}$$

where the third equality follows from applying $d\rho d\rho=d\phi d\phi=d\theta d\theta=0$, the fourth equality follows from swapping differential forms where necessary, and the fifth equality follows from canceling the top two terms in the line above. Finally, multiplying on the right by dz and applying Equations (5.4) gives

$$dxdydz=(dxdy)(dz)=(\rho\sin^2\phi\;d\rho d\theta+\rho^2\sin\phi\cos\phi\;d\phi d\theta)(\cos\phi\;d\rho-\rho\sin\phi\;d\phi)$$

$$
\begin{aligned}
&= \rho \sin^2\phi \cos\phi \; d\rho d\theta d\rho - \rho^2 \sin^3\phi \; d\rho d\theta d\phi + \rho^2 \sin\phi \cos^2\phi \; d\phi d\theta d\rho - \\
&\quad \rho^3 \sin^2\phi \cos\phi \; d\phi d\theta d\phi \\
&= -\rho^2 \sin^3\phi \; d\rho d\theta d\phi + \rho^2 \sin\phi \cos^2\phi \; d\phi d\theta d\rho \\
&= \rho^2 \sin^3\phi \; d\rho d\phi d\theta - \rho^2 \sin\phi \cos^2\phi \; d\phi d\rho d\theta \\
&= \rho^2 \sin^3\phi \; d\rho d\phi d\theta + \rho^2 \sin\phi \cos^2\phi \; d\rho d\phi d\theta \\
&= (\rho^2 \sin\phi)(\sin^2\phi + \cos^2\phi) \; d\rho d\phi d\theta \\
&= \rho^2 \sin\phi \; d\rho d\phi d\theta,
\end{aligned}
$$

where the third equality follows from the fact that $d\rho d\theta d\rho = d\phi d\theta d\phi = 0$, and the remainder follows from swapping differential forms where necessary. Again note that the term $\rho^2 \sin\phi$ pops out at the end of the computation. We call $\rho^2 \sin\phi \; d\rho d\phi d\theta$ the *volume form* of $\mathbb{R}^3$ in spherical coordinates.