NOTES ON L. C. EVANS: *PARTIAL DIFFERENTIAL EQUATIONS*

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Notes on chapters 5 and 6 of *Partial Differential Equations* by L. C. Evans evans:pde [1].

CONTENTS

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1. PRELIMINARIES

1.1. **Partial Differential Equations.**

In this section we introduce the definition of a partial differential equation.

A *partial differential equation* is an equation that involves an unknown function of two or more variables and partial derivatives of this unknown function with respect to its independent variables.

For the remainder of this section, fix an open subset $\Omega \subset \mathbb{R}^n$.

Let us recall that if $u : \Omega \to \mathbb{R}$ is any function, we write

$$u(x) = u(x_1, \dots, x_n), \quad x \in \Omega.$$

We may also consider vector–valued functions. If $\mathbf{u} : \Omega \to \mathbb{R}^m$, we write

$$\mathbf{u}(x) = (u^1(x), \dots, u^m(x)), \quad x \in \Omega.$$

The function u^i is the $i-$th component of $\mathbf{u}$, $i = 1, \dots, m$.

We will need some notation for the partial derivatives of a sufficiently smooth function $u : \Omega \to \mathbb{R}$. Let us first recall that

$$\mathcal{C}^k(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is } k-\text{ times continuously differentiable}\}$$

and

$$\mathcal{C}^k(\overline{\Omega}) = \big\{u \in \mathcal{C}^k(\Omega) : D^\alpha u \text{ is uniformly continuous on bounded subsets of } \Omega, \\ \text{for all } |\alpha| \le k\big\}.$$

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See below for a review of multi–index notation. Recall also that we denote the partial derivative of u with respect to x_i by

$$u_{x_i}(x) = \frac{\partial u}{\partial x_i}(x) = \lim_{h\to 0} \frac{u(x+he_i) - u(x)}{h},$$

provided that this limit exists.

Next we recall multi–index notation.

Definition (Multi–index, Order of Multi–index)**.** *A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each component α_i, $i = 1, \dots, n$ is a nonnegative integer, is called a* **multi–index** *of* **order**

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

The purpose of multi–index notation is to write the partial derivatives of u more cleanly. Given a multi–index α, we define

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(x).$$

ex:1.1-1 **Example 1.1.1.** *If $u : \mathbb{R}^2 \to \mathbb{R}$ is given by $u(x) := x_1^2 e^{3x_2}$ and $\alpha = (1, 2)$, then*

$$D^\alpha u(x) = \partial_{x_1} \partial_{x_2}^2 u(x) = 18 x_1 e^{3x_2}, \quad x \in \mathbb{R}^2.$$

Next, if $k \geq 0$ is an integer, we write

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\},$$

the set of all partial derivatives of order k. If we assign some ordering to the various partial derivatives, we can also regard $D^k u(x)$ as a point in $\mathbb{R}^{n^k}$. We define

$$|D^k u| := \left(\sum_{|\alpha|=k} |D^\alpha u|^2 \right)^{\frac{1}{2}}.$$

Before giving the definition of a partial differential equation, we first give some special cases of multi–index notation that will be useful to us later. If $k = 1$, we regard the elements of Du as being arranged in a vector, called the *gradient vector* of u :

$$Du := (u_{x_1}, \dots, u_{x_n}).$$

Therefore Du gives a vector field in $\mathbb{R}^n$. If $k = 2$, we regard the elements of $D^2 u$ as being arranged in a matrix, called the *Hessian matrix* of u :

$$D^2 u := \begin{pmatrix} \partial_{x_1}^2 u & \cdots & \partial_{x_1 x_n} u \\ \vdots & \ddots & \vdots \\ \partial_{x_n x_1} u & \cdots & \partial_{x_n}^2 u \end{pmatrix}.$$

Therefore $D^2 u \in \mathbb{S}^n$, the space of real symmetric $n \times n$ matrices.

Definition (Partial Differential Equation)**.** *An expression of the form*

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0, \quad x \in \Omega \tag{1.1.1}$$ {eq:1.1-1

*is called a $k-$***th order partial differential equation***, where*

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given and

$$u : \Omega \to \mathbb{R}$$

is the unknown.

Note that in (1.1.1) we do not actually assert that $u \in \mathcal{C}^k(\Omega)$, even though we are considering the $k-$th order partial derivatives of u. We will see why this makes sense in §2.2.

We say that we *solve* the PDE (1.1.1) if we find all functions u satisfying (1.1.1), possibly among only those functions satisfying certain auxiliary boundary conditions on some part Γ of the boundary $\partial\Omega$ of Ω. By "finding the solutions," we mean, ideally, obtaining simple, explicit solutions, or, failing that, deducing the existence and other properties of solutions.

We may also categorize PDE by their linearity/nonlinearity.

Definition (Linear PDE, Homogeneous)**.** *The PDE (1.1.1) is called* **linear** *if it has the form*

$$\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u = f(x) \tag{1.1.2}$$

for given functions $a_\alpha (|\alpha| \le k), f : \Omega \to \mathbb{R}$. Moreover, the linear PDE (1.1.2) is called **homogeneous** *if $f \equiv 0$.*

Definition (Semilinear PDE)**.** *The PDE (1.1.1) is called* **semilinear** *if it has the form*

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0. \tag{1.1.3}$$

Definition (Quasilinear PDE)**.** *The PDE (1.1.1) is called* **quasilinear** *if it has the form*

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0. \tag{1.1.4}$$

Definition (Fully Nonlinear PDE)**.** *The PDE (1.1.1) is called* **fully nonlinear** *if it depends nonlinearly upon the highest order derivatives.*

Lastly, we may also consider *systems* of PDE, which we briefly present here. A system of partial differential equations is, informally speaking, a collection of several PDE for several unknown functions.

Definition (System of PDE)**.** *An expression of the form*

$$\mathbf{F}(D^k\mathbf{u}(x), D^{k-1}\mathbf{u}(x), \ldots, D\mathbf{u}(x), \mathbf{u}(x), x) = \mathbf{0}, \quad x \in \Omega, \tag{1.1.5}$$

*is called a $k-$***th order system of partial differential equations***, where*

$$\mathbf{F} : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$$

is given and

$$\mathbf{u} : \Omega \to \mathbb{R}^m, \quad \mathbf{u} = (u^1, \ldots, u^m)$$

is the unknown.

Here we are supposing that the system comprises the same number m of scalar equations as unknowns $(u^1, \ldots, u^m)$. This is the most common circumstance, although other systems may have fewer or more equations than unknowns. Systems are classified in the obvious way as being linear, semilinear, etc., as above. Note again that we do not assert that $\mathbf{u} \in \mathcal{C}^k(\Omega; \mathbb{R}^m)$.

1.2. **Convolution and Mollification.** We next want to introduce tools that will allow us to construct smooth approximations to certain functions. These will be important in the proofs of the Sobolev space approximation theorems in §2.3.

Definition (Ω_ϵ)**.** *If $\Omega \subset \mathbb{R}^n$ is open and $\epsilon > 0$, we write*

$$\Omega_\epsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}.$$

Thus we see that Ω_ϵ is the set of all points in Ω that are "away from the boundary."

Definition (Standard Mollifier)**.** *We define the* **standard mollifier** $\eta \in \mathcal{C}^\infty(\mathbb{R}^n)$ *by*

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where the constant $C > 0$ is selected so that $\int_{\mathbb{R}^n} \eta \, dx = 1$.

Definition (Mollifier)**.** *For each $\epsilon > 0$, we define the* **mollifier**

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right).$$

Note that for each $\epsilon > 0$, the functions η_ϵ belong to $\mathcal{C}^\infty(\mathbb{R}^n)$ and satisfy

$$\int_{\mathbb{R}^n} \eta_\epsilon \, dx = 1, \quad \operatorname{supp}(\eta_\epsilon) \subset B(0, \epsilon).$$

Definition (Mollification)**.** *If $f \in L^1_{\text{loc}}(\Omega)$, we define its mollification*

$$f_\epsilon := \eta_\epsilon * f \quad \text{in } \Omega_\epsilon.$$

Recalling the definition of the convolution of two functions, that is,

$$f_\epsilon(x) = \int_\Omega \eta_\epsilon(x-y) f(y) \, dy = \int_{B(0,\epsilon)} \eta_\epsilon(y) f(x-y) \, dy$$

for all $x \in \Omega_\epsilon$. Recall also that

$$\operatorname{supp}(f_\epsilon) = \operatorname{supp}(\eta_\epsilon + f) \subset \operatorname{supp}(\eta_\epsilon) + \operatorname{supp}(f).$$

Before presenting several properties of the mollification of a locally integrable function f, the intuitive idea is to "smooth out" or "average out" f. Note that we only require that f is locally integrable on Ω, and these functions may be very irregular. When we take the convolution of f with η_ϵ and take the limit as $\epsilon \to 0$, we are "smoothing out" sharp features of f while still remaining close – in a certain, specific sense, as we will see in the following theorem – to the original (nonsmoooth) function f.

t1.2-1 **Theorem 1.2.1** (Properties of Mollifiers)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $f \in L^1_{\text{loc}}(\Omega)$. Then*

(i) $f_\epsilon \in \mathcal{C}^\infty(\Omega_\epsilon)$;
(ii) $f_\epsilon \to f$ $\mathcal{L}^n$*–a.e. as* $\epsilon \to 0$;
(iii) If $f \in \mathcal{C}(\Omega)$, then $f_\epsilon \to f$ uniformly on compact subsets of Ω;
(iv) If $1 \leq p < +\infty$ and $f \in L^p_{\text{loc}}(\Omega)$, then $f_\epsilon \to f$ in $L^p_{\text{loc}}(\Omega)$.

Proof.

i. Fix $x \in \Omega_\epsilon$, $i \in \{1, \dots, n\}$, and h so small that $x + he_i \in \Omega_\epsilon$. Since $\operatorname{supp}(\eta_\epsilon) \subset B(0, \epsilon)$, there exists an open set $U \subset\subset \Omega$ such that $\eta_\epsilon(x + he_i - y)$ and $\eta_\epsilon(x - y)$ both vanish for all $y \in \Omega \setminus U$. Thus

$$\begin{aligned}
\frac{f_\epsilon(x + he_i) - f_\epsilon(x)}{h} &= \int_\Omega (\eta_\epsilon(x + he_i - y) - \eta_\epsilon(x - y)) f(y)\, dy \\
&= \frac{1}{\epsilon^n} \int_\Omega \frac{1}{h} \left[\eta\left(\frac{x + he_i - y}{h} \right) - \eta\left(\frac{x - y}{h} \right) \right] f(y)\, dy \\
&= \frac{1}{\epsilon^n} \int_U \frac{1}{h} \left[\eta\left(\frac{x + he_i - y}{h} \right) - \eta\left(\frac{x - y}{h} \right) \right] f(y)\, dy.
\end{aligned}$$

Since

$$\lim_{h \to 0} \frac{\eta\left(\frac{x + he_i - y}{\epsilon} \right) - \eta\left(\frac{x - y}{\epsilon} \right)}{h} = \frac{1}{\epsilon} \eta_{x_i} \left(\frac{x - y}{\epsilon} \right) = \epsilon^n \partial_{x_i} \eta_\epsilon(x - y).$$

uniformly on U, Lebesgue's Dominated Convergence Theorem shows that the partial derivative $\partial_{x_i} f_\epsilon(x)$ and equals

$$\partial_{x_i} f_\epsilon(x) \overset{L.D.C.}{=} \int_\Omega \partial_{x_i} \eta_\epsilon(x - y) f(y)\, dy$$

for all $x \in \Omega_\epsilon$. Repeating this argument as necessary shows that $D^\alpha f_\epsilon(x)$ exists, and

$$D^\alpha f_\epsilon(x) = \int_\Omega D^\alpha \eta_\epsilon(x - y) f(y)\, dy, \quad x \in \Omega$$

for each multi–index α. This proves assertion (i).

ii. By Lebesgue's Differentiation Theorem,

$$\lim_{r \to 0} ⨍_{B(x,r)} |f(y) - f(x)|\, dy = 0 \tag{1.2.1}$$ {eq:1.2-1

for $\mathcal{L}^n$–a.e. $x \in \Omega$. Fix such a point $x \in \Omega$. Then, noting that $\eta(\frac{x-y}{\epsilon}) \le Ce$ for all $y \in B(x, \epsilon)$,

$$\begin{aligned}
|f_\epsilon(x) - f(x)| &= \left| \int_B (x, \epsilon) \eta_\epsilon(x - y) f(y)\, dy - f(x) \int_{B(x,\epsilon)} \eta_\epsilon(x - y)\, dy \right| \\
&= \left| \int_{B(x,\epsilon)} \eta_\epsilon(x - y)[f(y) - f(x)]\, dy \right| \\
&\le \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\left(\frac{x - y}{\epsilon} \right) |f(y) - f(x)|\, dy \\
&\le C ⨍_{B(x,\epsilon)} |f(y) - f(x)|\, dy \to 0
\end{aligned}$$

as $\epsilon \to 0$, by (1.2.1), as required. This proves assertion (ii).

iii. Assume now that $f \in \mathcal{C}(\Omega)$. Choose $U \subset\subset \Omega$, and then choose any $U \subset\subset V \subset\subset \Omega$ and note that f is uniformly continuous on V. Thus the limit in (1.2.1) holds uniformly for $x \in U$. Consequently the calculation in part (ii) implies that $f_\epsilon \to f$ uniformly on U.

iv. Next assume that $1 \le p < +\infty$ and $f \in L^p_{\text{loc}}(\Omega)$. Choose an open set $U \subset\subset \Omega$ and, as above, choose an open set V so that $U \subset\subset V \subset\subset \Omega$. We claim that for sufficiently small

$\epsilon > 0$,

$$\|f_\epsilon\|_{L^p(U)} \leq \|f\|_{L^p(W)}. \tag{1.2.2}$$

To see this, we note that if $1 \leq p < +\infty$ and $x \in V$, then

$$\begin{aligned}
|f_\epsilon(x)| &= \left| \int_{B(x,\epsilon)} \eta_\epsilon(x-y) f(y)\, dy \right| \\
&\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)^{1-\frac{1}{p}} \eta_\epsilon(x-y)^{\frac{1}{p}} |f(y)|\, dy \\
&\leq \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)\, dy \right)^{1-\frac{1}{p}} \cdot \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y) |f(y)|^p\, dy \right)^{\frac{1}{p}},
\end{aligned}$$

where we have used Hölder's inequality with conjugate exponents p and $\frac{p}{p-1}$ on the RHS. Since $\int_{B(x,\epsilon)} \eta_\epsilon(x-y) = 1$, this inequality implies that

$$\begin{aligned}
\int_U |f_\epsilon(x)|^p\, dx &\leq \int_U \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y) |f(y)|^p\, dy \right) dx \\
&\leq \int_V |f(y)|^p \left(\int_{B(y,\epsilon)} \eta_\epsilon(x-y)\, dx \right) dy \\
&= \int_W |f(y)|^p\, dy
\end{aligned}$$

for all $\epsilon > 0$ sufficiently small. This is (1.2.2).

v. Now fix $U \subset\subset V \subset\subset \Omega$, $\delta > 0$, and choose $g \in \mathcal{C}(V)$ such that

$$\|f - g\|_{L^p(V)} < \delta. \tag{1.2.3}$$

Then by Minkowski's inequality and (1.2.2), we have

$$\begin{aligned}
\|f^\epsilon - f\|_{L^p(U)} &\leq \|f_\epsilon - g_\epsilon\|_{L^p(U)} + \|g_\epsilon - g\|_{L^p(U)} + \|g - f\|_{L^p(U)} \\
&\leq 2\|f - g\|_{L^p(V)} + \|g_\epsilon - g\|_{L^p(U)} \\
&\leq 2\delta + \|g_\epsilon - g\|_{L^p(U)}.
\end{aligned}$$

Since $g_\epsilon \to g$ uniformly on U by assertion (iii) of the theorem, we have

$$\limsup_{\epsilon \to 0} \|f_\epsilon - f\|_{L^p(U)} \leq 2\delta,$$

as required, since U and V were arbitrary. The proof is complete. □

Remark.

(i) If $f \in L^p_{\text{loc}}(\Omega)$, inequality (1.2.2) asserts that mollification reduces the $L^p_{\text{loc}}-$norm of f.

(ii) Notice that Theorem 1.2.1(iv) **does not** *hold in the case $p = +\infty$. This is because $\mathcal{C}(\Omega)$ is not dense in $L^\infty(\Omega)$, so we may not necessarily be able to find $g \in \mathcal{C}(\Omega)$ such that (1.2.3) holds.*

1.3. **Boundaries.** In this section we give the definition of a $\mathcal{C}^k-$ smooth boundary and discuss its properties. We let Ω be an open, bounded subset of $\mathbb{R}^n$, and $k \in \mathbb{N}$.

Definition ($\mathcal{C}^k$ Boundary)**.** *We say that the boundary $\partial\Omega$ of Ω is $\mathcal{C}^k$ if for each point $x_0 \in \partial\Omega$ there exist $r > 0$ and a $\mathcal{C}^k$ function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon relabeling and reorienting the coordinate axes if necessary, we have*

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$

We sometimes also write $\partial\Omega \in \mathcal{C}^k$.

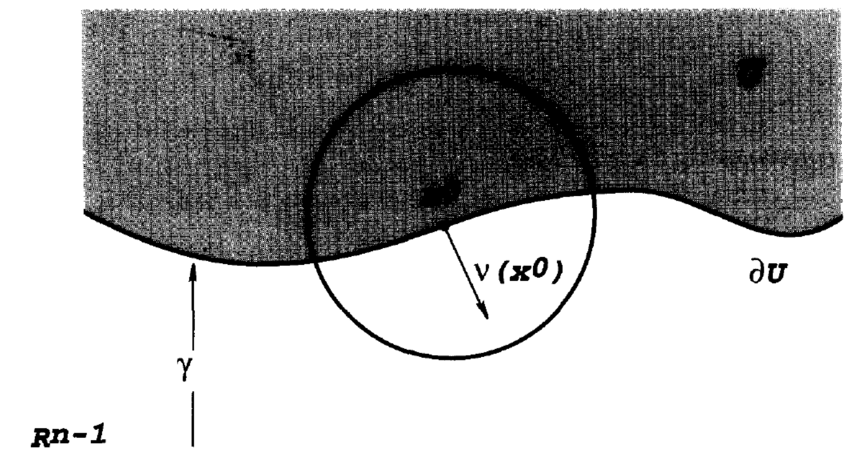

FIGURE 1.3.1. The Boundary of Ω.

Intuitively, the definition of a $\mathcal{C}^k$ boundary $\partial\Omega$ states that $\partial\Omega$ is given by the graph of a $\mathcal{C}^k(\mathbb{R}^{n-1})$ function γ and is one–sided, that is, no part of the domain Ω can lie on both sides of any part of the boundary $\partial\Omega$.

ex1.3-1 **Example 1.3.1** (A Smooth Boundary)**.** *Any open ball $B(x, r)$ in $\mathbb{R}^n$ has a smooth ($\mathcal{C}^\infty$) boundary for all $n \geq 1$.*

We show this for the open unit ball $\Omega := B(0, 1)$ in $\mathbb{R}^2$. To see this, fix any point $(x_0, y_0) \in \partial\Omega$, choose $r = 1 > 0$, and reorient the coordinate axes so that $(x_0, y_0) = (0, -1)$. Then

$$\begin{aligned}\Omega \cap B((x_0, y_0), r) &= \Omega \cap B((0, -1), 1) \\ &= \left\{(x, y) \in \mathbb{R}^2 : -\frac{\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2},\ -\sqrt{1 - x^2} < y < +\sqrt{1 - x^2} - 1\right\}\end{aligned}$$

Define then the function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(x) := -\sqrt{1 - x^2}.$$

Clearly $\gamma \in \mathcal{C}^\infty(-1, 1)$. It remains to show that

$$\Omega_1 := \Omega \cap B((0, -1), 1) = \{(x, y) \in B((0, -1), 1) : y > \gamma(x)\} =: \Omega_2.$$

It is obvious that $\Omega_1 \subseteq \Omega_2$. For the reverse inclusion, let $(x, y) \in \Omega_2$. First note that if $|x| \geq \frac{\sqrt{3}}{2}$, then

$$y > \gamma(x) = -\sqrt{1 - x^2} \geq -\sqrt{1 - \frac{3}{4}} = -\frac{1}{2}.$$

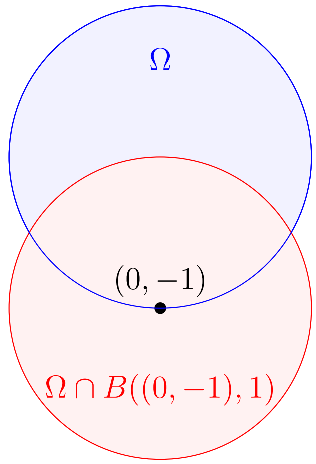

FIGURE 1.3.2. Ω and $\Omega \cap B((0,-1),1)$.

But this is impossible since $(x,y) \in B((0,-1),1)$, *and*

$$\sqrt{x^2+(y+1)^2} \geq \sqrt{\frac{3}{4}+\left(1-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1.$$

So then $-\frac{-\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2}$. *Now if* $y \geq \sqrt{1-x^2}-1$, *we obtain another contradiction, for*

$$\sqrt{x^2+(y+1)^2} \geq \sqrt{x^2+(1-x^2)} = 1.$$

Thus $\Omega_2 \subseteq \Omega_1$, *so that* $\Omega_1 = \Omega_2$, *as required.*

ex1.3-2 **Example 1.3.2** (A Nonsmooth Boundary). *We now give an example of a boundary that is not* $\mathcal{C}^1$. *Define* $\Omega \subset \mathbb{R}^2$ *by*

$$\Omega := B(0,1) \setminus [0,1).$$

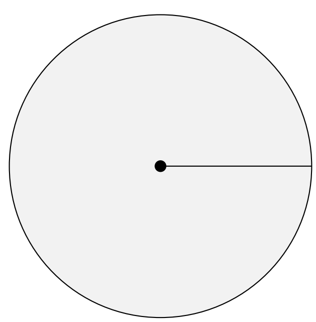

FIGURE 1.3.3. $B(0,1) \setminus [0,1)$.

To see that $\partial\Omega$ *is not* $\mathcal{C}^1$, *first note that* $x_0 := 0 \in \partial\Omega$. *Let* $r > 0$. *Then*

$$\Omega \cap B(0,r) = B(0,\min\{1,r\}) \setminus [0,\min\{1,r\}).$$

Suppose by contradiction that $\partial\Omega$ *is* $\mathcal{C}^1$. *Put* $r_0 := \frac{\min\{1,r\}}{2}$ *and choose any sequence* $\{x_m\}_{m=1}^{+\infty}$ *such that* $x_m > 0$, $x_m \to 0$, *and* $(x_m, r_0) \in \Omega \cap B(0,r)$ *for all* $m \in \mathbb{N}$. *Since* $\partial\Omega$ *is* $\mathcal{C}^1$, *there exists* $\gamma \in \mathcal{C}^1(\mathbb{R})$ *such that*

$$y > \gamma(x) \quad \textit{for all } (x,y) \in B(0,r).$$

By continuity,

$$r_0 > \lim_{m\to+\infty} \gamma(x_m) = \gamma(0).$$

But note also that $(x_m, -r_0) \in \Omega \cap B(0,r)$ *for all* $m \in \mathbb{N}$. *Thus*

$$-r_0 > \lim_{m\to+\infty} \gamma(x_m) = \gamma(0),$$

a contradiction, because $r_0 > 0$.

The definition of an "analytic boundary" is the obvious one.

Definition (Analytic Boundary). *We say that the boundary* $\partial\Omega$ *of* Ω *is* **analytic** *if for each point* $x_0 \in \partial\Omega$ *there exist* $r > 0$ *and an analytic function* $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ *such that, upon relabeling and reorienting the coordinate axes if necessary, we have*

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, \dots, x_{n-1})\}.$$

We will often have occasion to consider the outer unit normal when applying the Gauss–Green theorem or the divergence theorem.

Definition (Outer Unit Normal). *If* $\partial\Omega$ *is* $\mathcal{C}^1$, *then along* $\partial\Omega$ *is defined the* **outward pointing unit normal vector field**

$$\nu = (\nu^1, \dots, \nu^n).$$

Definition (Outer Normal Derivative). *Let* $u \in \mathcal{C}^1(\Omega)$. *We call*

$$\frac{\partial u}{\partial \nu} := \nu \cdot Du$$

the **outward normal derivative** *of* u.

We will often consider BVPs with Dirichlet boundary conditions, that is, equations of the form

$$\begin{cases} F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0, & x \in \Omega, \\ u|_{\partial\Omega} = g, \end{cases}$$

where g is some given function. In this case the outer unit normal along $\partial\Omega$ is

$$\frac{Du}{|Du|} = \frac{1}{|Du|}\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right),$$

provided that $Du \not\equiv 0$.

Flattening the Boundary. We will frequently need to change coordinates near a point of $\partial\Omega$ so as to "flatten out" the boundary. To be specific, fix $x_0 \in \partial\Omega$, and choose $r > 0$ and $\gamma \in \mathcal{C}^k(\mathbb{R}^{n-1})$ as above. Define then

$$\begin{cases} y_i = x_i =: \Phi^i(x), \quad i = 1, \dots, n-1, \\ y_n = x_n - \gamma(x_1, \dots, x_{n-1}) =: \Phi^n(x), \end{cases}$$

and write

$$y = \Phi(x).$$

Similarly, we set

$$\begin{cases} x_i = y_i =: \Psi^i(y), \quad i = 1, \dots, n-1, \\ x_n = y_n + \gamma(y_1, \dots, y_{n-1}) =: \Psi^n(y), \end{cases}$$

and write

$$x = \Psi(y).$$

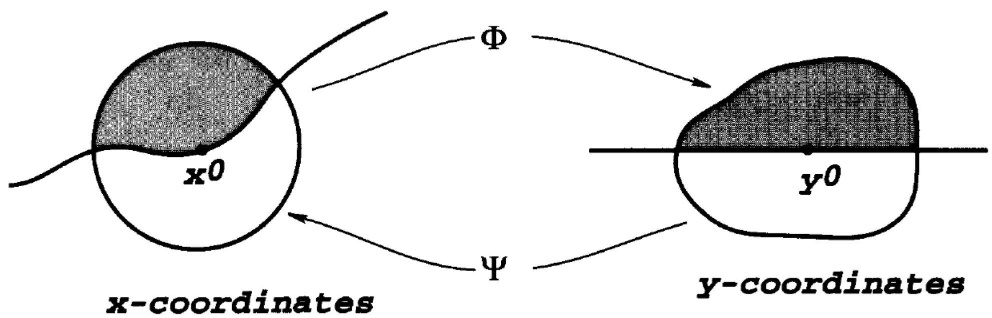

FIGURE 1.3.4. Flatting out the boundary.

Then $\Phi = \Psi^{-1}$, and the mapping $x \mapsto \Phi(x) = y$ "straightens out $\partial\Omega$" near x_0. Observe also that

$$D\Phi = \begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ -\gamma_{x_1} & \cdots & -\gamma_{x_{n-1}} & 1 \end{pmatrix}$$

and

$$D\Psi = \begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \gamma_{y_1} & \cdots & \gamma_{y_{n-1}} & 1 \end{pmatrix},$$

so that $\det D\Phi = \det D\Psi = 1$.

2. Sobolev Spaces

In this section we mostly develop the theory of *Sobolev spaces*, which we will see to be the proper settings in which to apply ideas of functional analysis when considering PDE.

Keeping in mind eventual applications to wide classes of PDEs (we want a theory that deals with linear/nonlinear elliptic, parabolic, and hyperbolic equations), we sketch out here our overall point of view. The intention is to take various specific PDEs and recast them abstractly as operators acting on appropriate normed linear spaces. We can symbolically write

$$A : X \to Y,$$

where the operator A describes the structure of the PDEs, including possibly boundary conditions, and X, Y are normed linear spaces of functions. The advantage of this formulation is that once our PDE problem has been interpreted in this form, we often can employ the general principles of functional analysis to study the properties (including solvability, existence, and uniqueness) of various equations involving the differential operator A. We will see that the most difficult work is not so much the invocation of functional analysis, but finding the "right" function spaces X and Y and the "right" differential operators A. Sobolev spaces are designed specifically to make this choices work out nicely.

As mentioned above, Sobolev spaces are useful for studying linear elliptic, parabolic, and hyperbolic PDE, as well as nonlinear elliptic and parabolic PDE.

2.1. **Hölder Spaces.**

Before studying Sobolev spaces, we first consider the simpler Hölder spaces.

Throughout this section, we assume that $\Omega \subseteq \mathbb{R}^n$ is open and $0 < \gamma \leq 1$.

We first recall the definition of Lipschitz continuity:

Definition (Lipschitz Continuity). *A function $u : \Omega \to \mathbb{R}$ is said to be* **Lipschitz continuous** *if there exists some constant $C > 0$ such that*

$$|u(x) - u(y)| \leq C|x - y|, \quad x, y \in \Omega. \tag{2.1.1}$$

Note that if (2.1.1) holds, then clearly u is continuous, and more importantly, (2.1.1) provides a modulus of continuity. It turns out to be useful to consider also functions satisfying a variant of (2.1.1).

Definition (Hölder Continuity). *A function $u : \Omega \to \mathbb{R}^n$ is said to be* **Hölder continuous with exponent** γ *if there exists $C > 0$ such that*

$$|u(x) - u(y)| \leq C|x - y|^\gamma, \quad x, y \in \Omega. \tag{2.1.2}$$

Let us recall here a simple but important definition from functional analysis.

Definition (Norm). *Let X be a linear space. A function $\|\cdot\| : X \to [0, +\infty)$ is called a* **norm** *on X if the following three conditions hold:*

(i) $\|u\| = 0$ *if and only if* $u = 0$;
(ii) $\|\gamma u\| = |\gamma|\|u\|$ *for all* $u \in X$ *and* $\gamma \in \mathbb{R}$ *(or* $\mathbb{C}$*);*
(iii) $\|u + v\| \leq \|u\| + \|v\|$ *for all* $u, v \in X$ *(triangle inequality).*

Definition ($\|\cdot\|_{\mathcal{C}(\overline{\Omega})}$)**.** *If $u : \Omega \to \mathbb{R}$ is bounded and continuous, we write*

$$\|u\|_{\mathcal{C}(\overline{\Omega})} := \sup_{x\in\Omega} |u(x)|.$$

Definition ($\gamma-$th Hölder Seminorm $|\cdot|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})}$)**.** *The $\gamma-$th Hölder seminorm of a function $u : \Omega \to \mathbb{R}$ is*

$$|u|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|} \right\}.$$

Note that $|\cdot|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})}$ is indeed only a seminorm, and not a norm, as any nonzero constant function $u : \Omega \to \mathbb{R}$ satisfies $|u|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} = 0$.

Definition ($\gamma-$th Hölder Norm $\|\cdot\|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})}$)**.** *The $\gamma-$th Hölder norm of a function $u : \Omega \to \mathbb{R}$ is*

$$\|u\|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} := \|u\|_{\mathcal{C}(\overline{\Omega})} + |u|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})}.$$

Definition (Hölder Space $\mathcal{C}^{k,\gamma}(\overline{\Omega})$)**.** *The Hölder space*

$$\mathcal{C}^{k,\gamma}(\overline{\Omega})$$

consists of all functions $u \in \mathcal{C}^k(\overline{\Omega})$ for which the norm

$$\|u\|_{\mathcal{C}^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{\mathcal{C}(\overline{\Omega})} + \sum_{|\alpha|=k} |D^\alpha u|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} \tag{2.1.3}$$ {eq:2.1-3

is finite.

Note that the space $\mathcal{C}^{k,\gamma}(\overline{\Omega})$ consists of all functions $u : \Omega \to \mathbb{R}$ that are $k-$times continuously differentiable and whose $k-$th partial derivatives are bounded and Hölder continuous with exponent γ. These functions are well–behaved, and furthermore we want to show that $\mathcal{C}^{k,\gamma}(\overline{\Omega})$ is a *Banach space*.

Definition (Banach Space)**.** *A* **Banach space** *is a normed linear space which is complete.*

t2.1-1 **Theorem 2.1.1.** *The Hölder space $\mathcal{C}^{k,\gamma}(\overline{\Omega})$ is a Banach space.*

Proof. It is clear that $\|\cdot\|_{\mathcal{C}^{k,\gamma}(\overline{\Omega})}$ is a norm on $\mathcal{C}^{k,\gamma}(\overline{\Omega})$.

It remains to show that $\mathcal{C}^{k,\gamma}(\overline{\Omega})$ is complete. Let $\{u_m\}_{m=1}^{+\infty} \subset \mathcal{C}^{k,\gamma}(\overline{\Omega})$ be a Cauchy sequence. Recalling that $\mathcal{C}^k(\overline{\Omega})$ is a Banach space under the norm

$$\|u\|_{\mathcal{C}^k(\overline{\Omega})} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{\mathcal{C}(\overline{\Omega})}$$

and $\mathcal{C}^{k,\gamma}(\overline{\Omega}) \subset \mathcal{C}^k(\overline{\Omega})$, define $u : \Omega \to \mathbb{R}$ by

$$u := \lim_{m\to+\infty} u_m \quad \text{in } \mathcal{C}^k(\overline{\Omega}). \tag{2.1.4}$$ {eq:2.1-4

We must first show that $u \in \mathcal{C}^{k,\gamma}(\overline{\Omega})$. Fix a multi–index α with $|\alpha| = k$. Note that by eq:2.1-4 (2.1.4),

$$D^\alpha u_m \to D^\alpha u \quad \text{uniformly on } \Omega.$$

Thus, for any $x, y \in \Omega$, $x \neq y$ and $m \in \mathbb{N}$,

$$\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} \leq \frac{|D^\alpha u(x) - D^\alpha u_m(x)|}{|x-y|^\gamma} + \frac{|D^\alpha u_m(x) - D^\alpha u_m(y)|}{|x-y|^\gamma} +$$

$$\begin{aligned} &\frac{|D^\alpha u_m(y) - D^\alpha u(y)|}{|x-y|^\gamma} \\ &\leq \frac{|D^\alpha u(x) - D^\alpha u_m(x)|}{|x-y|^\gamma} + |D^\alpha u_m|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} + \frac{|D^\alpha u_m(y) - D^\alpha u(y)|}{|x-y|^\gamma}. \end{aligned}$$

By the uniform convergence, we may choose $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$,

$$\frac{|D^\alpha u(x) - D^\alpha u_{m_0}(x)|}{|x-y|^\gamma} + \frac{|D^\alpha u_{m_0}(y) - D^\alpha u(y)|}{|x-y|^\gamma} \leq 1.$$

Hence,

$$\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\alpha} \leq 1 + |D^\alpha u_{m_0}|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})}.$$

This is for all $x, y \in \Omega$, $x \neq y$ so that taking the supremum on the LHS over all $x, y \in \Omega$, $x \neq y$, and multi–indices $|\alpha| = k$ gives

$$|D^\alpha u|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} \leq 1 + |D^\alpha u_{m_0}|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} < +\infty,$$

as required.

Finally, we show that $\{D^\alpha u_m\}_{m=1}^{+\infty}$ converges to $D^\alpha u$ in the Hölder seminorm. Choose any $x, y \in \Omega$, $x \neq y$, and $m \in \mathbb{N}$. Then

$$\begin{aligned} \frac{|(D^\alpha u_m - D^\alpha u)(x) - (D^\alpha u_m - D^\alpha u)(y)|}{|x-y|^\gamma} &= \frac{|D^\alpha u_m(x) - D^\alpha u(x) - D^\alpha u_m(y) + D^\alpha u(y)|}{|x-y|^\gamma} \\ &\leq \frac{|D^\alpha u_m(x) - D^\alpha u(x)|}{|x-y|^\gamma} + \frac{|D^\alpha u_m(y) - D^\alpha u(y)|}{|x-y|^\gamma} \\ &= \lim_{l \to +\infty} \left\{ \frac{|D^\alpha u_l(x) - D^\alpha u_m(x)|}{|x-y|^\gamma} + \frac{|D^\alpha u_l(y) - D^\alpha u_m(y)|}{|x-y|^\gamma} \right\}, \end{aligned}$$

and, since $\{D^\alpha u_m\}_{m=1}^{+\infty}$ is Cauchy in $\mathcal{C}^{0,\gamma}(\overline{\Omega})$, the RHS vanishes by taking the limit as $m \to +\infty$. Hence

$$D^\alpha u_m \to D^\alpha u \quad \text{in } \mathcal{C}^{k,\gamma}(\overline{\Omega}),$$

as required. The proof is complete. □

2.2. **Sobolev Spaces.** Hölder spaces as introduced in §5.1 are unfortunately not often suitable settings for PDE theory, as we generally cannot make good enough analytic estimates to demonstrate that the solutions we construct actually belong to such spaces. What are needed are some other kinds of spaces, containing less smooth functions. In practice we must strike a balance, by designing spaces comprising functions which have some, but not too great, smoothness properties.

What we will end up defining is a space of functions $u : \Omega \to \mathbb{R}^n$ with k "weak derivatives." That is, the functions belonging to this space may not have derivatives in the classical sense, but have "derivatives" that behave nicely with respect to integration against a certain class of functions.

. . . .

2.2.1. *Weak Derivatives.* As mentioned, we start by weakening the notion of partial derivatives.

Definition ($\mathcal{C}_c^\infty(\Omega)$, Test Function)**.** *We denote by*

$$\mathcal{C}_c^\infty(\Omega)$$

the space of all infinitely differentiable functions $\phi : \Omega \to \mathbb{R}$ with compact support in Ω. We call a function $\phi \in \mathcal{C}_c^\infty(\Omega)$ a **test function***.*

Motivation for Definition of Weak Derivative. Assume that we are given a function $u \in \mathcal{C}^1(\Omega)$. Then if $\phi \in \mathcal{C}_c^\infty(\Omega)$, we see from the integration by parts formula that

$$\int_\Omega u\phi_{x_i}\,dx = -\int_\Omega u_{x_i}\phi\,dx + \int_{\partial\Omega} u v \nu^i\,dS = -\int_\Omega u_{x_i}\phi\,dx, \quad i = 1, \ldots, n, \tag{2.2.1}$$

where the boundary term vanishes because ϕ has compact support in Ω. More generally now, if k is a positive integer, $u \in \mathcal{C}^k(\Omega)$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi–index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n \le k$, then

$$\int_\Omega u D^\alpha \phi\,dx = (-1)^{|\alpha|}\int_\Omega D^\alpha u \phi\,dx. \tag{2.2.2}$$

Note that the equality in (2.2.2) holds because

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \phi,$$

and we may apply formula (2.2.1) $|\alpha|$ times.

We next assume that formula (2.2.2) holds for some function $u : \Omega \to \mathbb{R}$ and every test function ϕ. Note that in (2.2.2) we required that u be $k-$times continuously differentiable, and we consider now if this requirement may be weakened, that is, if (2.2.2) may still be true even if u is not $\mathcal{C}^k-$smooth. Note that the LHS of (2.2.2) makes sense if u is only locally integrable, as ϕ and all its derivatives have compact support in Ω. The problem is the RHS: if u is not $\mathcal{C}^k-$smooth, then the expression "$D^\alpha u$" has no obvious meaning. We resolve this issue by formulating the definition of a "weak derivative" of u, that is, a locally integrable function $v : \Omega \to \mathbb{R}$ for which formula (2.2.2) holds, with v in place of $D^\alpha u$.

Definition (Weak Derivative)**.** *Suppose that $u, v \in L^1_{\text{loc}}(\Omega)$ and α is a multi–index. We say that v is the $\alpha^{th}-$weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided that

$$\int_\Omega u D^\alpha \phi\,dx = (-1)^{|\alpha|}\int_\Omega v\phi\,dx \tag{2.2.3}$$

holds for all test functions $\phi \in \mathcal{C}_c^\infty(\Omega)$.

That is, if we are given a function $u \in L^1_{\text{loc}}(\Omega)$ and if there happens to exist a function $v \in L^1_{\text{loc}}(\Omega)$ which satisfies (2.2.3) for all test functions ϕ, we then say that $D^\alpha u = v$ in the weak sense. If on the other hand there does not exist such a function v, then evidently u does not possess an $\alpha^{\text{th}}-$weak partial derivative.

Recall that if u has a classical derivative, it is clear that this derivative is unique. We show that the same is true for weak derivatives, at least up to sets of measure zero.

l2.2-1 **Lemma 2.2.1** (Uniqueness of Weak Derivatives). *Suppose that $u \in L^1_{\text{loc}}(\Omega)$ possesses a weak α^{th} partial derivative v. Then v is uniquely defined up to a set of measure zero.*

Proof. Suppose that $v, w \in L^1_{\text{loc}}(\Omega)$ satisfy

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega v\phi \, dx = (-1)^{|\alpha|} \int_\Omega w\phi \, dx$$

for all $\phi \in \mathcal{C}_c^\infty(\Omega)$. Then evidently

$$\int_\Omega (v - w)\phi \, dx = 0 \tag{2.2.4}$$ {eq:2.2-4

for all $\phi \in \mathcal{C}_c^\infty(\Omega)$, so that $v = w$ a.e. The proof is complete. □

ex:2.2-1 **Example 2.2.1.** *Let $n = 1$, $\Omega = (0, 2)$, and*

$$u(x) := \begin{cases} x, & 0 < x \leq 1, \\ 1, & 1 \leq x < 2. \end{cases}$$

Note that u is not differentiable at $x = 1$ in the classical sense. However, differentiating the piecewise components of u, we might expect that

$$u'(x) = \begin{cases} 1, & 0 < x \leq 1, \\ 0, & 1 < x < 2 \end{cases}$$

to be the weak derivative of u.

To verify this, choose any $\phi \in \mathcal{C}_c^\infty(\Omega)$. By (2.2.3), we must show that

$$\int_0^2 u\phi' \, dx = -\int_0^2 u'\phi \, dx.$$

Integrating by parts and using the fact that ϕ vanishes at $x = 0$ and $x = 2$, we calculate

$$\begin{aligned} \int_0^2 u\phi' \, dx &= \int_0^1 x\phi' \, dx + \int_1^2 \phi' \, dx \\ &= \phi(1) - \phi(0) - \int_0^1 \phi \, dx + \phi(2) - \phi(1) \\ &= -\int_0^1 \phi \, dx \\ &= -\int_0^2 u'\phi \, dx, \end{aligned}$$

as required.

ex:2.2-2 **Example 2.2.2.** *Let $n = 1$, $\Omega = (0, 2)$, and*

$$u(x) := \begin{cases} x, & 0 < x \leq 1, \\ 2, & 1 < x < 2. \end{cases}$$

Note again that u is not differentiable at $x = 1$. Based on the procedure in Example 2.2.1, we might expect that

$$v(x) := \begin{cases} 1, & 0 < x \leq 1, \\ 0, & 1 < x < 2 \end{cases}$$

is the weak derivative of u. However, let us show that u does not have a weak derivative.

For motivation, let us first choose a test function $\phi \in \mathcal{C}_c^\infty(\Omega)$ and observe that

$$\begin{aligned}
\int_0^2 u\phi' \, dx &= \int_0^1 x\phi' \, dx + \int_1^2 2\phi' \, dx \\
&= \phi(1) - \phi(0) - \int_0^1 \phi \, dx + 2\phi(2) - 2\phi(1) \\
&= -\int_0^1 v\phi \, dx - \phi(1) \\
&= -\int_0^1 v\phi \, dx - \delta_1(\phi).
\end{aligned}$$

Intuitively, the problem is that the Dirac delta distribution is not a function – it is a distribution.

To verify that u possesses no weak derivative, suppose by contradiction that there exists $v \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_0^2 u\phi' \, dx = -\int_0^2 v\phi \, dx \tag{2.2.5}$$

{eq:2.2-5

for all $\phi \in \mathcal{C}_c^\infty(\Omega)$. Then

$$\begin{aligned}
-\int_0^2 v\phi \, dx &= \int_0^2 u\phi' \, dx \\
&= \int_0^1 x\phi' \, dx + 2\int_1^2 \phi' \, dx \\
&= \phi(1) - \phi(1) - \int_0^1 \phi \, dx + 2\phi(2) - \phi(1) \\
&= -\int_0^1 \phi \, dx - \phi(1).
\end{aligned} \tag{2.2.6}$$

{eq:2.2-6

Choose then a sequence $\{\phi_m\}_{m=1}^{+\infty} \subset \mathcal{C}_c^\infty(\Omega)$ such that

$$0 \le \phi_m \le 1, \quad \phi_m(1) = 1, \quad \phi_m(x) \to 0 \textit{ for all } x \neq 1.$$

But then replacing ϕ with ϕ_m in (2.2.6) and taking the limit as $\phi \to +\infty$, we find

eq:2.2-6

$$1 = \lim_{m\to+\infty} \phi_m(1) = \lim_{m\to+\infty} \left[\int_0^2 v\phi_m \, dx - \int_0^1 \phi_m \, dx\right] \overset{L.D.C.}{=} 0,$$

a contradiction.

. . . .

2.2.2. *Definition of Sobolev Spaces.* Fix $1 \le p \le +\infty$ and let k be a nonnegative integer. With the definition of a weak derivative in mind, we now define certain function spaces comprised of functions which have weak derivatives of various orders lying in various L^p spaces.

Definition (Sobolev Space $W^{k,p}(\Omega)$). *The* **Sobolev space**

$$W^{k,p}(\Omega)$$

consists of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that for each multi–index α with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

Note that choosing $|\alpha| = 0$ implies that $u \in L^p(\Omega)$.

Remark.

(i) If $p = 2$, we write

$$H^k(\Omega) := W^{k,2}(\Omega), \quad k \in \mathbb{N}_0.$$

The letter H is used, since, as we will see later, $H^k(\Omega)$ is a Hilbert space. Note also that $H^0(\Omega) = L^2(\Omega)$.

(ii) We identify functions in $W^{k,p}(\Omega)$ which coincide $\mathcal{L}^n$−a.e.

Definition ($\|\cdot\|_{W^{k,p}(\Omega)}$). *If $u \in W^{k,p}(\Omega)$, we define the $W^{k,p}(\Omega)$−norm of u by*

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha u|^p \, dx \right)^{\frac{1}{p}}, & 1 \le p < +\infty, \\ \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = +\infty. \end{cases}$$

Notice that the $W^{k,p}(\Omega)$−norm of a function $u \in W^{k,p}(\Omega)$ is the Minkowski norm of the $L^p(\Omega)$−norms of all weak derivatives of u if $1 \le p < +\infty$, and the sum of all $L^\infty(\Omega)$−norms of all weak derivatives of u if $p = +\infty$.

Another choice of norm on $W^{k,p}(\Omega)$ is given by

$$\|u\|'_{W^{k,p}(\Omega)} := \begin{cases} \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)}, & 1 \le p < +\infty, \\ \max_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = +\infty. \end{cases}$$

It may be shown that the norms $\|\cdot\|_{W^{k,p}(\Omega)}$ and $\|\cdot\|'_{W^{k,p}(\Omega)}$ are equivalent.

Definition (Convergence in $W^{k,p}(\Omega)$). *Let $\{u_m\}_{m=1}^{+\infty}, u \in W^{k,p}(\Omega)$. We say that u_m* **converges** *to u in $W^{k,p}(\Omega)$, and write*

$$u_m \to u \quad \text{in } W^{k,p}(\Omega),$$

provided that

$$\lim_{m \to +\infty} \|u_m - u\|_{W^{k,p}(\Omega)} \to 0.$$

Definition (Convergence in $W^{k,p}_{\mathrm{loc}}(\Omega)$). *Let $\{u_m\}_{m=1}^{+\infty}, u \in W^{k,p}(\Omega)$. We say that u_m* **converges** *to u in $W^{k,p}_{\mathrm{loc}}(\Omega)$, and write*

$$u_m \to u \quad \text{in } W^{k,p}_{\mathrm{loc}}(\Omega),$$

provided that

$$u_m \to u \quad \text{in } W^{k,p}(U)$$

for all $U \subset\subset \Omega$.

Definition ($W^{k,p}_0(\Omega)$). *We define $W^{k,p}_0(\Omega)$ to be the closure of $\mathcal{C}^\infty_c(\Omega)$ in $W^{k,p}(\Omega)$, that is,*

$$W^{k,p}_0(\Omega) := W^{k,p} \cap \overline{\mathcal{C}^\infty_c(\Omega)}.$$

Thus $u \in W_0^{k,p}(\Omega)$ if and only if there exists a sequence $\{u_m\}_{m=1}^{+\infty} \subset \mathcal{C}_c^\infty(\Omega)$ such that $u_m \to u$ in $W^{k,p}(\Omega)$. We will see later that we can think of $W_0^{k,p}(\Omega)$ functions as $W^{k,p}(\Omega)$ functions whose first $k-1$ partial derivatives "vanish at the boundary of Ω" (specifically, we will see that they have zero *trace*), that is, all functions $u \in W^{k,p}(\Omega)$ such that

$$\text{“}D^\alpha u = 0 \text{ on } \partial\Omega\text{” for all } |\alpha| \le k-1.$$

Remark (Notation)**.** *We will write*

$$H_0^k(\Omega) := W_0^{k,2}(\Omega).$$

In fact if $n = 1$ and Ω is an open interval in $\mathbb{R}$, then $u \in W^{1,p}(\Omega)$ if and only if u is equal $\mathcal{L}^1-$a.e. to an absolutely continuous function whose ordinary derivative (which exists $\mathcal{L}^1-$a.e.) belongs to $L^p(\Omega)$. This simple characterization is however only available for $n = 1$. In general a function can belong to a Sobolev space and yet be discontinuous and/or unbounded.

ex:2.2-3 **Example 2.2.3.** *Take $\Omega := B(0,1)$, the open unit ball in $\mathbb{R}^n$, and let*

$$u(x) := |x|^{-\alpha}, \quad x \in \Omega,\ x \neq 0.$$

We consider the values of $\alpha > 0$, n, and p for which u belongs to $W^{1,p}(\Omega)$. Note first that u is smooth away from $x = 0$, with

$$\frac{\partial u}{\partial x_i}(x) = -\frac{\alpha}{2|x|^{\alpha-2}} \cdot 2x_i = \frac{-\alpha x_i}{|x|^{\alpha+2}}, \quad x \neq 0,$$

and thus

$$|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}}.$$

Now fix $\phi \in \mathcal{C}_c^\infty(\Omega)$ and $\epsilon > 0$. Then integration by parts gives

$$\int_{\Omega\setminus B(0,\epsilon)} u\phi_{x_i}\, dx = -\int_{\Omega\setminus B(0,\epsilon)} u_{x_i}\phi\, dx + \int_{\partial B(0,\epsilon)} u\phi\nu^i\, dS,$$

where $\nu := (\nu^1, \dots, \nu^n)$ denotes the inward pointing unit normal on $\partial B(0,\epsilon)$. Now if $\alpha + 1 < n$, then $|Du(x)| \in L^1(\Omega)$. In this case

$$\left|\int_{\partial B(0,\epsilon)} u\phi\nu^i\, dS\right| \le \int_{\partial B(0,\epsilon)} |u\phi\nu^i|\, d\mathcal{H}^{n-1} \le \|\phi\|_{L^\infty(\Omega)}\epsilon^{-\alpha}\int_{\partial B(0,\epsilon)} d\mathcal{H}^{n-1}$$
$$\le C\epsilon^{n-1-\alpha} \quad \to 0 \text{ as } \epsilon \to 0$$

since $n - 1 - \alpha > 0$. Thus

$$\int_\Omega u\phi_{x_i}\, dx = -\int_\Omega u_{x_i}\phi\, dx$$

for all $\phi \in \mathcal{C}_c^\infty(\Omega)$, provided that $0 \le \alpha < n-1$. Furthermore $|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \in L^p(\Omega)$ if and only if $(\alpha+1)p < n$. Consequently $u \in W^{1,p}(\Omega)$ if and only if $\alpha < \frac{n-p}{p}$, since $\frac{n-p}{p} \le n-1$. In particular $u \notin W^{1,p}(\Omega)$ for any $p \ge n$.

ex:2.2-4 **Example 2.2.4.** *Let $\{r_k\}_{k=1}^{+\infty}$ be a countable, dense subset of $\Omega = B(0,1)$. Write*

$$u(x) := \sum_{k=1}^{+\infty} \frac{1}{2^k}|x - r_k|^{-\alpha}, \quad x \in \Omega.$$

Then $u \in W^{1,p}(\Omega)$ for $\alpha < \frac{n-p}{p}$. If $0 < \alpha < \frac{n-p}{p}$, we see that u belongs to $W^{1,p}(\Omega)$ and yet is unbounded on each open subset of Ω.

The last example shows that although a function u belonging to a Sobolev space possesses certain smoothness properties, it can still be rather badly behaved in other ways.

. . . .

2.2.3. *Elementary Properties.* Next we show certain properties of weak derivatives. Note here that these rules are obvious for smooth functions, but functions belonging to Sobolev spaces are not necessarily regular, and thus we have to rely only on the definition of weak derivatives.

t2.2-1 **Theorem 2.2.1** (Properties of Weak Derivatives). *Assume that $u, v \in W^{k,p}(\Omega)$ and let α be a multi–index with $|\alpha| \leq k$. Then*

(i) $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all multi–indices α, β with $|\alpha| + |\beta| \leq k$;

(ii) For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$;

(iii) If U is an open subset of Ω, then $u \in W^{k,p}(U)$;

(iv) If $\zeta \in \mathcal{C}_c^\infty(\Omega)$, then $\zeta u \in W^{k,p}(\Omega)$ and

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \quad \textit{(Leibniz's formula)}, \tag{2.2.7}$$ {eq:2.2-7

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

Proof.

i. The first assertion in (i) is clear by choosing any multi–index β such that $|\alpha| + |\beta| = k$ and applying the definition of weak derivatives and $W^{k,p}(\Omega)$.

To prove the second assertion, fix $\phi \in \mathcal{C}_c^\infty(\Omega)$. Then $D^\beta \phi \in \mathcal{C}_c^\infty(\Omega)$, and so

$$\begin{aligned}
\int_\Omega D^\alpha u D^\beta \phi \, dx &= (-1)^{|\alpha|} \int_\Omega u D^{\alpha+\beta} \phi \, dx \\
&= (-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \int_\Omega (D^{\alpha+\beta} u) \phi \, dx \\
&= (-1)^{|\beta|}(-1)^{2|\alpha+\beta|} \int_\Omega u D^{\alpha+\beta} \phi \, dx \\
&= (-1)^{|\beta|} \int_\Omega u D^{\alpha+\beta} \phi \, dx.
\end{aligned}$$

Thus $D^\beta(D^\alpha u) = D^{\alpha+\beta}u$ in the weak sense. Similarly $D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ in the weak sense, which proves (i).

ii. The first assertion in (ii) simply states that $W^{k,p}(\Omega)$ is a real linear space, which is clear from linearity of the integral. The second assertion also follows by linearity of the integral as follows: for any $\phi \in \mathcal{C}_c^\infty(\Omega)$, we have

$$\int_\Omega D^\alpha(\lambda u + \mu v)\phi \, dx = (-1)^{|\alpha|} \int_\Omega (\lambda u + \mu v) D^\alpha \phi \, dx$$

$$
\begin{aligned}
&= (-1)^{|\alpha|}\lambda \int_\Omega u D^\alpha \phi \, dx + (-1)^{|\alpha|}\mu \int_\Omega v D^\alpha \phi \, dx \\
&= \lambda \int_\Omega (D^\alpha u)\phi \, dx + \mu \int_\Omega (D^\alpha v)\phi \, dx \\
&= \int_\Omega (\lambda D^\alpha u + \mu D^\alpha v)\phi \, dx.
\end{aligned}
$$

Since this is for all $\phi \in \mathcal{C}_c^\infty(\Omega)$, (ii) follows.

iii. Assertion (iii) follows immediately by taking the restrictions of u and $D^\alpha u$ from Ω to U.

iv. To prove (2.2.7), we use induction on $|\alpha|$. First suppose that $|\alpha| = 1$. Choose any $\phi \in \mathcal{C}_c^\infty(\Omega)$. Then

$$
\begin{aligned}
\int_\Omega \zeta u D^\alpha \phi \, dx &= \int_\Omega u D^\alpha(\zeta\phi) - u(D^\alpha \zeta)\phi \, dx \\
&= -\int_\Omega (D^\alpha u \zeta + u D^\alpha \zeta)\phi \, dx.
\end{aligned}
$$

Thus $D^\alpha(\zeta u) = D^\alpha u \zeta + u D^\alpha \zeta$, as required. Note that on the LHS we have used the product rule on $\zeta\phi$ since we cannot necessarily apply the product rule on $u\zeta$.

Assume now for the induction hypothesis that $l < k$ and that formula (2.2.7) holds for all $|\alpha| \le l$ and all functions $\zeta \in \mathcal{C}_c^\infty(\Omega)$. Choose a multi–index α with $|\alpha| = l + 1$. Then $\alpha = \beta + \gamma$ for some $|\beta| = l$, $|\gamma| = 1$. Then for any $\phi \in \mathcal{C}_c^\infty(\Omega)$,

$$
\begin{aligned}
\int_\Omega \zeta u D^\alpha \phi \, dx &= \int_\Omega D^\beta(D^\gamma \phi) \, dx \\
&= (-1)^{|\beta|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\sigma \zeta D^{\beta-\sigma} u D^\gamma \phi \, dx
\end{aligned}
$$

(by the induction hypothesis)

$$
= (-1)^{|\beta|+|\gamma|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\gamma (D^\sigma \zeta D^{\beta-\sigma} u)\phi \, dx
$$

(by the induction hypothesis again)

$$
= (-1)^{|\alpha|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left[D^\rho \zeta D^{\alpha-\rho} u + D^\sigma \zeta D^{\alpha-\sigma} u \right] \phi \, dx
$$

(where $rho := \sigma + \gamma$)

$$
= (-1)^{|\alpha|} \int_\Omega \left[\sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} D^\sigma \zeta D^{\alpha-\sigma} u \right] \phi \, dx,
$$

since

$$\binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma} = \binom{\alpha}{\sigma}.$$

The proof is complete. □

Theorem t2.2-1 2.2.1 shows that many of the familiar rules of calculus apply to weak derivatives. The following theorem states that the Sobolev spaces are in fact Banach spaces.

t2.2-2 **Theorem 2.2.2.** *For each $k \in \mathbb{N}_0$ and $1 \le p \le +\infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Proof.

i. We first of all check that $\|\cdot\|_{W^{k,p}(\Omega)}$ is a norm. Clearly

$$\|\lambda u\|_{W^{k,p}(\Omega)} = |\lambda| \|u\|_{W^{k,p}(\Omega)}$$

for all $\lambda \in \mathbb{R}$, and

$$\|u\|_{W^{k,p}(\Omega)} = 0 \text{ if and only if } u = 0 \ \mathcal{L}^n - \text{a.e.}$$

Next assume that $u, v \in W^{k,p}(\Omega)$. Then if $1 \le p < +\infty$, Minkowski's inequality implies that

$$\begin{aligned}
\|u+v\|_{W^{k,p}(\Omega)} &= \left(\sum_{|\alpha| \le k} \|D^\alpha u + D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \\
&\overset{M.I.}{\le} \left(\sum_{|\alpha| \le k} \left(\|D^\alpha u\|_{L^p(\Omega)} + \|D^\alpha v\|_{L^p(\Omega)} \right)^p \right)^{\frac{1}{p}} \\
&\overset{M.I.}{\le} \left(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} + \left(\sum_{|\alpha| \le k} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \\
&= \|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}(\Omega)}.
\end{aligned}$$

The case $p = +\infty$ follows immediately from the fact that $\|\cdot\|_{L^\infty(\Omega)}$ is a norm. Thus $\|\cdot\|_{W^{k,p}(\Omega)}$ is in fact a norm on $W^{k,p}(\Omega)$.

ii. It remains to show that $W^{k,p}(\Omega)$ is complete. Assume that $\{u_m\}_{m=1}^{+\infty} \subset W^{k,p}(\Omega)$ is a Cauchy sequence. Then for each $|\alpha| \le k$, $\{D^\alpha u_m\}_{m=1}^{+\infty}$ is a Cauchy sequence in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete, for each multi–index α with $|\alpha| \le k$ there exist functions $u_\alpha \in L^p(\Omega)$ such that

$$D^\alpha u_m \to u_\alpha \quad \text{in } L^p(\Omega).$$

In particular, notice that

$$u_m \to u_{(0,\dots,0)} =: u \quad \text{in } L^p(\Omega).$$

iii. We now claim that

$$u \in W^{k,p}(\Omega), \quad D^\alpha u = u_\alpha, \quad |\alpha| \le k. \tag{2.2.8}$$ {eq:2.2-8

To show this, fix $\phi \in \mathcal{C}_c^\infty(\Omega)$. Then

$$\begin{aligned}
\int_\Omega u D^\alpha \phi \, dx &\overset{L.D.C.}{=} \lim_{m \to +\infty} \int_\Omega u_m D^\alpha \phi \, dx \\
&= \lim_{m \to +\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_m \phi \, dx
\end{aligned}$$

$$\overset{L.D.C.}{=} (-1)^{|\alpha|} \int_\Omega u_\alpha \phi \, dx.$$

Thus (2.2.8) is valid, which shows $u \in W^{k,p}(\Omega)$. But then $D^\alpha u_m \to D^\alpha u$ in $L^p(\Omega)$ for all multi–indices $|\alpha| \leq k$, so that $u_m \to u$ in $W^{k,p}(\Omega)$, as required. The proof is complete. □

2.3. **Approximation.** We want to avoid returning to the definition of weak derivatives in the proofs of future results. We want to develop some systematic procedures for approximating a function in a Sobolev space by smooth functions. The method of mollification in §1.2 provides a way to do this.

To be more precise, we want conditions that also us to approximate a function $u \in W^{k,p}(\Omega)$ by a sequence $\{u_m\}_{m=1}^{+\infty} \subset \mathcal{C}^\infty(\Omega)$. The advantage is this. Smooth functions have many “nice" properties, and being able to write a Sobolev function as the limit of smooth functions allow us to exploit these properties, so long as we may apply a suitable convergence theorem (usually Lebesgue’s Dominated Convergence Theorem) when we pass to the limit of integrals.

For the remainder of this section, fix $k \in \mathbb{N}$ and $1 \leq p < +\infty$. Recall that

$$\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}.$$

. . . .

2.3.1. *Interior Approximation by Smooth Functions.*

t2.3-1 **Theorem 2.3.1** (Local Approximation by Smooth Functions)**.** *Assume that $u \in W^{k,p}(\Omega)$ for some $1 \leq p < +\infty$, and let*

$$u_\epsilon := \eta_\epsilon * u \quad \textit{in } \Omega_\epsilon$$

be the mollification of u. Then

(i) $u_\epsilon \in \mathcal{C}^\infty(\Omega_\epsilon)$ *for all* $\epsilon > 0$;

(ii) $u_\epsilon \to u$ *in* $W^{k,p}_{\mathrm{loc}}(\Omega)$ *as* $\epsilon \to 0$.

Proof.

i. Assertion (i) is a direct consequence of Theorem 1.2.1(i).

(ii). We next claim that if $|\alpha| \leq k$, then

$$D^\alpha u_\epsilon = \eta_\epsilon * D^\alpha u \quad \text{in } \Omega_\epsilon, \tag{2.3.1}$$

{eq:2.3-1

that is, the ordinary $\alpha^{\text{th}}-$ partial derivative of the smooth function u_ϵ is the $\epsilon-$mollification of the $\alpha^{\text{th}}-$weak partial derivative of u. To see this, first notice that for any $x \in \Omega_\epsilon$,

$$\begin{aligned} D^\alpha u_\epsilon(x) &= D^\alpha_x \left(\int_\Omega \eta_\epsilon(x-y) u(y) \, dy \right) \\ &= \int_\Omega D^\alpha_x (\eta_\epsilon(x-y)) u(y) \, dy \\ &= \int_\Omega (-1)^{|\alpha|} D^\alpha_y (\eta_\epsilon(x-y)) u(y) \, dy \\ &= (-1)^{|\alpha|} \int_\Omega D^\alpha_y (\eta_\epsilon(x-y)) u(y) \, dy. \end{aligned}$$

Now for any fixed $x \in \Omega_\epsilon$, the function $\phi(y) := \eta_\epsilon(x-y)$ belongs to $\mathcal{C}_c^\infty(\Omega)$, for $\operatorname{supp}(\eta_\epsilon) \subset B(0,\epsilon)$. Consequently the definition of the $\alpha^{\text{th}}-$weak partial derivative of u implies that

$$\int_\Omega D_y^\alpha(\eta_\epsilon(x-y))u(y)\,dy = (-1)^{|\alpha|}\int_\Omega \eta_\epsilon(x-y)D_y^\alpha u(y)\,dy.$$

Thus

$$\begin{aligned} D^\alpha u_\epsilon(x) &= (-1)^{|\alpha|}\int_\Omega D_y^\alpha(\eta_\epsilon(x-y))u(y)\,dy \\ &= (-1)^{2|\alpha|}\int_\Omega \eta_\epsilon(x-y)D^\alpha u(y)\,dy \\ &= (\eta_\epsilon * D^\alpha u)(x). \end{aligned}$$

This proves (2.3.1).

iii. Now choose any open set $U \subset\subset \Omega$. By Theorem 1.2.1(iv) and (2.3.1), we have that

$$D^\alpha u_\epsilon \overset{(2.3.1)}{=} D^\alpha u_\epsilon \to D^\alpha u \quad \text{in } L^p(U)$$

as $\epsilon \to 0$, for each $|\alpha| \le k$. Consequently

$$\|u_\epsilon - u\|_{W^{k,p}(U)}^p = \sum_{|\alpha|\le k}\|D^\alpha u_\epsilon - D^\alpha u\|_{L^p(U)}^p \to 0$$

as $\epsilon \to 0$. This proves assertion (ii). The proof is complete. □

Note that we cannot expect Theorem 2.3.1 to hold in the case $p = +\infty$ since $\mathcal{C}(U)$ is not sense in $L^\infty(U)$ for any open set $U \subset \mathbb{R}^n$.

. . . .

2.3.2. *Global Apprixmation by Smooth Functions.* Next we show that we can find smooth functions which approximate a function in $W^{k,p}(\Omega)$ and not just in $W^{k,p}_{\text{loc}}(\Omega)$. Note in the following that we make no assumptions about the smoothness of $\partial\Omega$.

t2.3-2 **Theorem 2.3.2** (Global Approximation by Smooth Functions)**.** *Assume that Ω is bounded, and suppose as well that $u \in W^{k,p}(\Omega)$ for some $1 \le p < +\infty$. Then there exists a sequence $\{u_m\}_{m=1}^{+\infty} \subset \mathcal{C}^\infty(\Omega)\cap W^{k,p}(\Omega)$ such that*

$$u_m \to u \quad \textit{in } W^{k,p}(\Omega).$$

Note that we *do not* assert that $u_m \in \mathcal{C}^\infty(\overline{\Omega})$.

Before giving the proof of Theorem 2.3.2, we need a definition.

Definition (Partition of Unity)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\{U_i\}_{i=1}^{+\infty}$ be any open cover of Ω. A* **partition of unity of Ω subordinate to** *$\{U_i\}_{i=1}^{+\infty}$ is a set $\{\zeta_i\}_{i=1}^{+\infty} \subset \mathcal{C}^\infty(\Omega)$ such that*

(i) $0 \le \zeta_i(x) \le 1$ *for all* $x \in \Omega$;
(ii) $\zeta_i \in \mathcal{C}_c^\infty(U_i)$ *for all* $i \in \mathbb{N}$;
(iii) $\sum_{i=1}^{+\infty}\zeta_i(x) = 1$ *for all* $x \in \Omega$.

A partition of unity always exists whenever a space is Hausdorff and paracompact (actually, this implies that the space is normal). Recall also that every metric space is compact.

Proof.

i. Put

$$U_i := \left\{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{i} \right\}, \quad i \in \mathbb{N}.$$

Note that $\Omega = \bigcup_{i=1}^{+\infty} U_i$. Write $V_i := U_{i+3} \setminus \overline{U}_{i+1}$, and note that V_i is open for all $i \in \mathbb{N}$.

Choose also any open set $V_0 \subset\subset \Omega$ such that $\Omega = \bigcup_{i=0}^{+\infty} V_i$. Now let $\{\zeta_i\}_{i=0}^{+\infty}$ be a smooth partition of unity subordinate to the open sets $\{V_i\}_{i=0}^{+\infty}$, that is,

$$\begin{cases} 0 \le \zeta_i \le 1, \quad \zeta_i \in \mathcal{C}_c^\infty(V_i), \\ \sum_{i=0}^{+\infty} \zeta_i = 1 \quad \text{on } \Omega. \end{cases} \tag{2.3.2}$$

Next choose any function $u \in W^{k,p}(\Omega)$. Note that $\operatorname{supp}(\zeta_i u) \subset V_i$, and, by Theorem 2.2.1(iv), $\zeta_i u \in W^{k,p}(\Omega)$ for all $i \in \mathbb{N}_0$.

ii. Fix $\delta > 0$. Choose then $\epsilon_i > 0$ so small that $u_i := \eta_{\epsilon_i} * (\zeta_i u)$ satisfies

$$\begin{cases} \|u_i - \zeta_i u\|_{W^{k,p}(\Omega)} \le \frac{\delta}{2^{i+1}}, & i \in \mathbb{N}_0, \\ \operatorname{supp}(u_i) \subset W_i, & i \in \mathbb{N}, \end{cases} \tag{2.3.3}$$

where $W_i := U_{i+4} \setminus \overline{U}_i \supset V_i$, $i \in \mathbb{N}$. Note that such u_i exist by Theorem 2.3.1. Note also that $u_i \in \mathcal{C}^\infty(\Omega)$ for all $i \in \mathbb{N}_0$.

iii. Write $v := \sum_{i=0}^{+\infty} u_i$. Notice that $v \in \mathcal{C}^\infty(\Omega)$, since for each open set $U \subset\subset \Omega$ there are at most finitely many terms in the sum, since the sequence $\{W_i\}_{i=1}^{+\infty}$ is increasing. Since $u = \sum_{i=0}^{+\infty} \zeta_i u$, we have for each open set $U \subset\subset \Omega$ by (2.3.3)

$$\begin{aligned} \|v - u\|_{W^{k,p}(U)} &\le \sum_{i=0}^{+\infty} \|u_i - \zeta_i u\|_{W^{k,p}(U)} \\ &\le \delta \sum_{i=0}^{+\infty} \frac{1}{2^{i+1}} \\ &= \delta. \end{aligned}$$

Taking the supremum over all open sets $U \subset\subset \Omega$, we conclude that

$$\|v - u\|_{W^{k,p}(\Omega)} \le \delta,$$

which shows that $\mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$. The proof is complete. □

. . . .

2.3.3. *Global Approximation by Smooth Functions Up to the Boundary.* We now want to approximate a function $u \in W^{k,p}(\Omega)$ by functions belonging to $\mathcal{C}^\infty(\overline{\Omega})$ rather than only $\mathcal{C}^\infty(\Omega)$. Assume that Ω is also bounded. Recalling that $\mathcal{C}^\infty(\overline{\Omega})$ is then the collection of all $u \in \mathcal{C}^\infty(\Omega)$ such that $D^\alpha u$ is uniformly continuous on Ω for any multi–index α, this means that our approximating sequence is smooth up to $\partial\Omega$. Therefore, such an approximation requires some condition on $\partial\Omega$.

t2.3-3 **Theorem 2.3.3** (Global Approximation by Functions Smooth Up to the Boundary). *Assume that Ω is bounded, with $\partial\Omega \in \mathcal{C}^1$, and let $u \in W^{k,p}(\Omega)$ for some $1 \le p < +\infty$. Then there exists a sequence $\{u_m\}_{m=1}^{+\infty} \subset \mathcal{C}^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ such that*

$$u_m \to u \quad \textit{in } W^{k,p}(\Omega).$$

Proof.
i. Fix any point $x_0 \in \partial\Omega$. Since $\partial\Omega \in \mathcal{C}^1$, there exist a radius $r > 0$ and a function $\gamma \in \mathcal{C}^1(\mathbb{R}^{n-1};\mathbb{R})$ such that, upon relabeling and reorienting the coordinate axes if necessary, we have

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, \dots, x_{n-1})\}.$$

Put $U := \Omega \cap B(x_0, \frac{r}{2})$.
ii. Define the shifted point

$$x_\epsilon := x + \lambda\epsilon e_n, \quad x \in U, \quad \epsilon > 0,$$

and note that, for some fixed, sufficiently large $\lambda > 0$, the ball $B(x_\epsilon, \epsilon)$ lies in $\Omega \cap B(x_0, r)$ for all $x \in U$ and $\epsilon > 0$ sufficiently small.

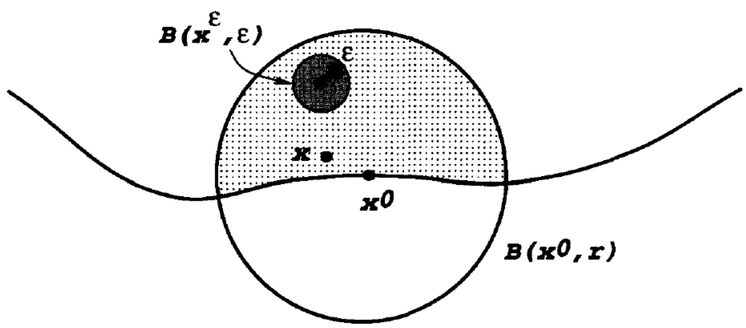

FIGURE 2.3.1. A ball around x_0 and x_ϵ.

Define also the function $u_\epsilon(x) := u(x_\epsilon)$, for all $x \in U$. Note that u_ϵ is defined on $U - \lambda\epsilon e_n$ and is the function u translated a distance $\lambda\epsilon$ in the e_n direction. The idea is to "leave room" to mollify within Ω.

iii. Since Ω is bounded, $\partial\Omega$ is compact, and we may cover $\partial\Omega$ with finitely many open sets $U_i := \Omega \cap B(x_i, \frac{r_i}{2})$, $i = 1, \dots, N$. Choose also any open set $U_0 \subset\subset \Omega$ such that $\Omega \subseteq \bigcup_{i=0}^N U_i$.

iv. Choose $\delta > 0$, and let $\{\zeta_i\}_{i=0}^N$ be a smooth partition of unity subordinate to the open sets $\{U_i\}_{i=0}^N$, that is,

$$\begin{cases} 0 \le \zeta_i \le 1, \quad \zeta_i \in \mathcal{C}_c^\infty(U_i), \\ \sum_{i=0}^N \zeta_i = 1 \quad \text{on } \Omega. \end{cases}$$

Note in particular that $u\zeta_0$ has compact support in $U_0 \subset\subset \Omega$. By mollifying $u\zeta_0$, we may find a function $v_0 \in \mathcal{C}_c^\infty(\Omega)$ such that

$$\|v_0 - u\zeta_0\|_{W^{k,p}(\Omega)} < \delta. \tag{2.3.4}$$ {eq:2.3-4

Since $v_0 \in \mathcal{C}_c^\infty(\Omega)$, then clearly $v_0 \in \mathcal{C}^\infty(\overline{\Omega})$.

v. Next, for each $\widetilde{\epsilon} > 0$, define $v^i_{\widetilde{\epsilon},\epsilon} := \eta_{\widetilde{\epsilon}} * u_\epsilon \zeta_i$. We claim that

$$\lim_{\epsilon\to 0}\lim_{\widetilde{\epsilon}\to 0} v^i_{\widetilde{\epsilon},\epsilon} = u\zeta_i \quad \text{in } W^{k,p}(\Omega). \tag{2.3.5}$$

First notice, since $\zeta_i \in \mathcal{C}^\infty_c(U_i)$, and recalling that

$$\operatorname{supp}(v^i_{\widetilde{\epsilon},\epsilon}) \subset \operatorname{supp}(\eta_{\widetilde{\epsilon}}) + \operatorname{supp}(u_\epsilon\zeta_i) \subset B(0,\epsilon) + \operatorname{supp}(u_\epsilon\zeta_i),$$

we may choose $\widetilde{\epsilon} > 0$ so small that $v^i_{\widetilde{\epsilon},\epsilon} \in \mathcal{C}^\infty_c(U_i)$. Now observe that

$$\|v^i_{\widetilde{\epsilon},\epsilon} - u\zeta_i\|_{W^{k,p}(\Omega)} \le \|v^i_{\widetilde{\epsilon},\epsilon} - u_\epsilon\zeta_i\|_{W^{k,p}(\Omega)} + \|u_\epsilon\zeta_i - u\zeta_i\|_{W^{k,p}(\Omega)}.$$

Taking the limit as $\widetilde{\epsilon} \to 0$, the first term on the RHS vanishes by reasoning similar to that of Theorem 2.3.1. For the second term, we have by Theorem 2.2.1

$$\begin{aligned}
\|u_\epsilon\zeta_i - u\zeta_i\|^p_{W^{k,p}(\Omega)} &= \sum_{|\alpha|\le k} \|D^\alpha(u_\epsilon\zeta_i - u\zeta_i)\|^p_{L^p(\Omega)} \\
&= \sum_{|\alpha|\le k} \|D^\alpha((u_\epsilon - u)\zeta_i)\|^p_{L^p(\Omega)} \\
&= \sum_{|\alpha|\le k} \left\| \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta(u_\epsilon - u) D^{\alpha-\beta}\zeta_i \right\|^p_{L^p(\Omega)} \\
&\le \sum_{|\alpha|\le k}\sum_{\beta\le\alpha} \binom{\alpha}{\beta} C \|D^\beta(u_\epsilon - u)\|^p_{L^p(\Omega)},
\end{aligned}$$

where the RHS follows because $\zeta_i \in \mathcal{C}^\infty_c(U_i)$. Since u_ϵ is a translation of u, $D^\beta u_\epsilon = (D^\beta u)_\epsilon$ for all multi–indices β, and translation is continuous in $L^p(\Omega)$, the RHS vanishes in the limit as $\epsilon \to 0$. This proves (v).

vi. By (2.3.4) and (2.3.5), there exist functions $\{v_0\}_{i=0}^N \subset \mathcal{C}^\infty(\overline{\Omega})$ such that

$$\|v_i - u\zeta_i\|_{W^{k,p}(\Omega)} < \delta$$

for each $i = 0,\dots,N$. Write $v := \sum_{i=0}^n v_i$. Clearly $v \in \mathcal{C}^\infty(\overline{\Omega})$. Since $u = \sum_{i=0}^N u\zeta_i$, then evidently

$$\begin{aligned}
\|v - u\|_{W^{k,p}(\Omega)} &\le \sum_{i=0}^N \|v_i - u\zeta_i\|_{W^{k,p}(\Omega)} \\
&< (N+1)\delta.
\end{aligned}$$

The proof is complete. □

2.4. **Extensions.** We want to extend functions in the Sobolev space $W^{1,p}(\Omega)$ to functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$, for $1 \le p < +\infty$.

This requires some caution. Notice for instance that simply extending a function $u \in W^{1,p}(\Omega)$ to be zero in $\mathbb{R}^n \setminus \Omega$ generally will not work, as this might create such a bad discontinuity along $\partial\Omega$ that the extended function no longer has a weak first partial derivative. We instead must formulate a way to extend functions $u \in W^{1,p}(\Omega)$ in a way which preserves the weak derivatives across $\partial\Omega$.

t2.4-1 **Theorem 2.4.1** (Extension Theorem). *Assume that $1 \le p < +\infty$, and also assume that Ω is bounded with $\partial\Omega \in \mathcal{C}^1$. Choose any open set $V \subset \mathbb{R}^n$ such that $\Omega \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n) \tag{2.4.1}$$

{eq:2.4-1

such that for each $u \in W^{1,p}(\Omega)$,

(i) $Eu = u$ $\mathcal{L}^n$–a.e. in Ω;

(ii) Eu has support within V;

(iii) There exists a constant $C > 0$, depending only on p, Ω, and V, such that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p(\Omega)}$.

Definition (Extension). *We call Eu an* **extension** *of u to $\mathbb{R}^n$.*

Remark. *The construction in the following proof is called a strong 1–extension operator, since the same construction works for all $1 \le p < +\infty$.*

Proof.

i. Fix $x_0 \in \partial\Omega$ and suppose first that

$$\partial\Omega \text{ is flat near } x_0, \text{ lying in the plane } \{x \in \mathbb{R}^n : x_n = 0\}. \tag{2.4.2}$$

{eq:2.4-2

Then we may assume that there exists an open ball B, with center x_0 and radius $r > 0$, such that

$$\begin{cases} B^+ := B \cap \{x \in \mathbb{R}^n : x_n \ge 0\} \subset \overline{\Omega}, \\ B^- := B \cap \{x \in \mathbb{R}^n : x_n \le 0\} \subset \mathbb{R}^n \setminus \Omega. \end{cases}$$

ii. Temporarily suppose also that $u \in \mathcal{C}^\infty(\overline{\Omega})$. Define then

$$\widetilde{u}(x) := \begin{cases} u(x), & x \in B^+, \\ -3u(x_1, \ldots, x_{n-1}, -x_n) + 4u(x_1, \ldots, x_{n-1}, -\frac{x_n}{2}), & x \in B^-. \end{cases} \tag{2.4.3}$$

{eq:2.4-3

We call $\widetilde{u}$ a *higher–order reflection* of u from B^+ to B^-. The number $-\frac{1}{2}$ associated with x_n in the second term can actually be replaced by any number $-\lambda$ with $0 < \lambda < 1$. Then the numbers -3 and 4 will have to be adjusted accordingly in the following steps.

iii. We claim that

$$\widetilde{u} \in \mathcal{C}^1(B). \tag{2.4.4}$$

{eq:2.4-4

Note that the only region of concern is $\{x_n = 0\}$. To see (2.4.4), define

$$\begin{cases} u^- := \widetilde{u}|_{B^-}, \\ u^+ := \widetilde{u}|_{B^+}. \end{cases}$$

We first show that

$$u^-_{x_n} = u^+_{x_n} \quad \text{on } \{x_n = 0\}. \tag{2.4.5}$$

{eq:2.4-5

We calculate, by (2.4.3),

$$u^+_{x_n} = u_{x_n}(x),$$

and

$$u^-_{x_n} = 3u_{x_n}(x_1, \ldots, x_{n-1}, -x_n) - 2u_{x_n}\left(x_1, \ldots, x_{n-1}, -\frac{x_n}{2}\right).$$

Thus

$$u^+_{x_n}|_{\{x_n=0\}} = u^-_{x_n}|_{\{x_n=0\}} = u_{x_n}(x_1, \ldots, x_{n-1}, 0),$$

which proves (2.4.5). Now since $u^+ = u^-$ on $\{x_n = 0\}$, it is also clear that

$$u^+_{x_i}|_{\{x_n=0\}} = u^-_{x_i}|_{\{x_n=0\}} = u_{x_i}(x_1, \dots, x_{n-1}, 0) \tag{2.4.6}$$ {eq:2.4-6

for $i = 1, \dots, n-1$. But then (2.4.5) and (2.4.6) imply that

$$D^\alpha u^+|_{\{x_n=0\}} = D^\alpha u^-|_{\{x_n=0\}}$$

for every multi–index $|\alpha| \le 1$, and so $\widetilde{u} \in \mathcal{C}^1(B)$. This proves (2.4.4).

iv. We next claim that

$$\|\widetilde{u}\|_{W^{1,p}(B)} \le C\|u\|_{W^{1,p}(B^+)}, \tag{2.4.7}$$ {eq:2.4-7

for some constant $C > 0$ which does not depend on u. To see this, we apply (2.4.3) to calculate

$$\begin{aligned}
\|\widetilde{u}\|^p_{L^p(B)} &= \int_B |\widetilde{u}|^p \, dx = \int_{B^+} |u^+|^p \, dx + \int_{B^-} |u^-|^p \, dx \\
&= \|u\|^p_{L^p(B^+)} + \int_{B^-} \left| -3u(x_1, \dots, x_{n-1}, -x_n) + 4u\left(x_1, \dots, x_{n-1}, -\frac{x_n}{2}\right)\right|^p \, dx \\
&\le \|u\|^p_{L^p(B^+)} + 4^p \int_{B^-} \left(|u(x_1, \dots, x_{n-1}, -x_n)| + \left| u\left(x_1, \dots, x_{n-1}, -\frac{x_n}{2}\right)\right|\right)^p \, dx \\
&\le \|u\|^p_{L^p(B^+)} + 4^p 2^p \left(\int_{B^-} |u(x_1, \dots, x_{n-1}, -x_n)|^p \, dx \; + \right. \\
&\qquad \left. \int_{B^-} \left| u\left(x_1, \dots, x_{n-1}, -\frac{x_n}{2}\right)\right|^p \, dx \right) \\
&= \|u\|^p_{L^p(B^+)} + 2^{3p} \left(\int_{B^+} |u(x_1, \dots, x_{n-1}, y_n)|^p \, dx_1 \cdots dx_{n-1} dy_n + \right. \\
&\qquad \left. 2\int_{B^+} |u(x_1, \dots, x_{n-1}, y_n)|^p \, dx_1 \cdots dx_{n-1} dy_n \right) \\
&= \|u\|^p_{L^p(B^+)} + 2^{3p} \cdot 3 \|u\|^p_{L^p(B^+)} \\
&\le C_p \|u\|^p_{L^p(B^+)},
\end{aligned}$$

where $C_p > 0$ is independent of u. Similarly, we can establish that

$$\|D\widetilde{u}\|^p_{L^p(B)} \le C_p \|Du\|^p_{L^p(B^+)}.$$

This proves (2.4.7).

v. Let us next consider the case that $\partial\Omega$ is not necessarily flat near x_0. Then, by §1.3, there exists a $\mathcal{C}^1$ mapping Φ, with inverse Ψ, such that Φ flattens out $\partial\Omega$ near x_0.

Write $y = \Phi(x)$, $x = \Psi(y)$, and $v(y) := u(\Psi(y))$. Choose a small ball $B = B(y_0, r)$ as in (i), that is, with B^+ contained in Ω in the positive y_n-coordinate plane. Using steps (i)–(iv) above, we extend v from B^+ to a function $\widetilde{v}$ defined on all of B, such that $\widetilde{v} \in \mathcal{C}^1(B)$, and in addition we have the estimate

$$\|\widetilde{v}\|_{W^{1,p}(B)} \le C\|v\|_{W^{1,p}(B^+)}.$$

Let $W := \Psi(B)$, and define $\widetilde{u}(x) := \widetilde{v}(\Psi(x))$. Then by the change of variables formula, recalling that $\det D\Psi = 1$, we have

$$\|\widetilde{u}\|^p_{W^{1,p}(W)} = \int_W |\widetilde{u}|^p + |D_x\widetilde{u}|^p \, dx$$

$$
\begin{aligned}
&= \int_{\Psi(B)} |\widetilde{u}|^p + |D\widetilde{u}|^p \, dx \\
&\leq \int_B (|\widetilde{v}|^p + C|D_y\widetilde{v}|^p) \cdot |\det D\Psi| \, dy \\
&\leq C \left(\int_B |\widetilde{v}|^p + |D_y\widetilde{v}|^p \, dy \right) \\
&= C\|\widetilde{v}\|^p_{W^{1,p}(B)} \\
&\leq C_2\|v\|^p_{W^{1,p}(B^+)} \\
&\leq C_3\|u\|^p_{W^{1,p}(W\cap\Omega)} \\
&\leq C_3\|u\|^p_{W^{1,p}(\Omega)}.
\end{aligned}
$$

Hence, we have obtained an extension $\widetilde{u}$ of u to W, with the estimate

$$\|\widetilde{u}\|^p_{W^{1,p}(W)} \leq \|u\|_{W^{1,p}(\Omega)}. \tag{2.4.8}$$ {eq:2.4-8}

vi. Since $\partial\Omega$ is compact, there exist finitely many points $x_i \in \partial\Omega$, open sets W_i, and extensions $\widetilde{u}_i$ of u to W_i, $i = 1, \ldots, N$, as above, such that $\partial\Omega \subset \bigcup_{i=1}^N W_i$. Choose any $W_0 \subset\subset \Omega$ so that $\Omega \subseteq \bigcup_{i=0}^N W_i$, and let $\{\zeta_i\}_{i=0}^N$ be a smooth partition of unity subordinate to the open sets $\{W_i\}_{i=0}^N$.

Define $\widehat{u} := \sum_{i=0}^N \zeta_i\widetilde{u}_i$, where $\widetilde{u}_0 = u$. Then by (2.4.8), we have that

$$
\begin{aligned}
\|\widehat{u}\|_{W^{1,p}(\mathbb{R}^n)} &\leq \sum_{i=0}^N \|\zeta_i\widetilde{u}_i\|_{W^{1,p}(\mathbb{R}^n)} \\
&= \sum_{i=0}^N \|\zeta_i\widetilde{u}_i\|_{W^{1,p}(W_i)} \\
&\leq C\sum_{i=0}^N \|\widetilde{u}_i\|_{W^{1,p}(W_i)} \\
&\leq C\|u\|_{W^{1,p}(\Omega)}.
\end{aligned}
$$

Hence we obtain the estimate

$$\|\widehat{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}, \tag{2.4.9}$$ {eq:2.4-9}

for some $C > 0$ independent of u.

Furthermore, by shrinking the radii $r_i > 0$ in each $B(x_i, r_i)$ if necessary, we can always ensure that $W_i \subset\subset V$ for any $\Omega \subset\subset V$. Therefore, we can assume that $\operatorname{supp}\widehat{u} \subset V$.

vii. Write $Eu := \widehat{u}$, and observe that the mapping $u \mapsto Eu$ is linear.

Recall that so far we have assumed that $u \in \mathcal{C}^\infty(\overline{\Omega})$. Suppose now only that $u \in W^{1,p}(\Omega)$. Since $\partial\Omega \in \mathcal{C}^1$, by Theorem 2.3.3 there exists a sequence $\{u_m\}_{m=1}^{+\infty} \subset \mathcal{C}^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ such that $u_m \to u$ in $W^{1,p}(\Omega)$. Now observe by (2.4.9) that

$$\|Eu_m - Eu_k\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|u_m - u_k\|_{W^{1,p}(\Omega)} \to 0$$

as $m, k \to +\infty$, so that $\{Eu_m\}_{m=1}^{+\infty}$ is a Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$. Since $W^{1,p}(\mathbb{R}^n)$ is a Banach space (cf. Theorem 2.2.2), there exists $u^* \in W^{1,p}(\mathbb{R}^n)$ such that $Eu_m \to u^*$ in $W^{1,p}(\mathbb{R}^n)$.

We claim that the limit u^* is independent of the choice of approximating sequence $\{u_m\}_{m=1}^{+\infty}$. To see this, suppose that $\{v_m\}_{m=1}^{+\infty} \subset \mathcal{C}^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is such that $v_m \to u$ in $W^{1,p}(\Omega)$ also. Then the sequence

$$\{u_1, v_1, u_2, v_2\} =: \{w_m\}_{m=1}^{+\infty}$$

is also Cauchy in $W^{1,p}(\Omega)$, and hence $Ew_m \to u^{**}$ in $W^{1,p}(\mathbb{R}^n$ for some $u^{**} \in W^{1,p}(\mathbb{R}^n)$. But, since any subsequence of a convergent subsequence has the same limit,

$$u^* = \lim_{m\to+\infty} Eu_m = \lim_{m\to+\infty} Ew_m = u^{**} \quad \text{in } W^{1,p}(\mathbb{R}^n),$$

as required.

Finally, we define $Eu := u^*$, so that $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ is a linear operator. It remains only to show that E is bounded. Since $u_m \in \mathcal{C}^\infty(\overline{\Omega})$, by (2.4.9), we have the estimate

$$\|Eu_m\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u_m\|_{W^{1,p}(\Omega)}.$$

Passing to the limit as $m \to +\infty$, we obtain

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} = \|u^*\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u^*\|_{W^{1,p}(\Omega)},$$

so that $E \in \mathcal{L}(W^{1,p}(\Omega), W^{1,p}(\mathbb{R}^n))$. The proof is complete. □

Remark.

(i) *Assume now that $\partial\Omega$ is $\mathcal{C}^2$. Then Φ and Ψ are $\mathcal{C}^2$ maps, but for $u \in \mathcal{C}^\infty(\overline{\Omega})$, Eu as constructed in steps (iii) and (iv) is not in general in $\mathcal{C}^2(B)$. However, it may be shown that $Eu \in W^{2,p}(B)$. Assuming this, we see that all future steps will follow immediately, and we have the bound*

$$\|Eu\|_{W^{2,p}(B)} \le C\|u\|_{W^{2,p}(B)}.$$

As in the proof, we consequently derive the estimate

$$\|Eu\|_{W^{2,p}(\mathbb{R}^n)} \le C\|u\|_{W^{2,p}(\Omega)}, \tag{2.4.10}$$

{eq:2.4-1

provided that $\partial\Omega$ is $\mathcal{C}^2$. Again, the constants C depend only on Ω, V, n, and p, but not u. Consequently $E \in \mathcal{L}(W^{2,p}(\Omega), W^{2,p}(\mathbb{R}^n))$.

(ii) *The above construction does* **not** *provide us with an extension for the Sobolev spaces $W^{k,p}(\Omega)$, if $k \ge 2$. However, it is true that there exists an extension operator $E \in \mathcal{L}(W^{k,p}(\Omega), W^{k,p}(\mathbb{R}^n))$. The proof requires a more complicated higher–order reflection technique as outlined below.*

To extend $W^{k,p}(\Omega)$ for $k > 2$ and $\partial\Omega \in \mathcal{C}^k$, let $x' := (x_1, \ldots, x_{n-1})$, and write

$$\widetilde{u}(x', x_n) := \sum_{i=1}^{k+1} c_i u\left(x', -\frac{x_n}{i}\right), \quad x_n < 0.$$

In order to maintain $\mathcal{C}^k$ continuity at $x_n = 0$, we need to obtain c_i, $i = 0, \ldots, k+1$, from

$$\sum_{i=1}^{k+1} c_i \left(\frac{-1}{i}\right)^m = 1, \quad \textit{for } m = 0, 1, \ldots, k.$$

Note that this is in the form $A\mathbf{c} = \mathbf{b}$, where A is a Vandermonde matrix, and hence A^{-1} exists.

References

1. L. C. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, 2010.