Elementary Differential Equations

MATH 2410Q Notes

Based On: Dennis G. Zill, *A first course in differential equations with modeling applications*, 12th ed., Cengage Learning, Boston, MA, 2024.

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Chapter 1

Introduction to differential equations

1.1 Definitions and terminology

Learning objectives

1. Define and classify differential equations according to **type, order, and linearity.**
2. Define the concept of the **solution** to a differential equation; understand that solutions always come specified with an **interval of existence**.
3. Understand the difference between an **ordinary differential equation** and **initial value problem**.
4. Be able to verify **solutions** to differential equations.
5. Interpret and sketch **integral curves** of solutions to **ordinary differential equations**.

Motivation and introduction

Recall that the exponential function $y = e^x$ satisfies $y' = e^x$. Hence, we can write

$$y' = y. \tag{1.1.1}$$

Now imagine that I give you (1.1.1) (without any prior knowledge of what y is) and tell you to solve for y. These are the kinds of problems you will be dealing with in this course. In fact, (1.1.1) is our first example of an *ordinary differential equation* (ODE). At first, it might look simple to solve: all we have to do is remember which function is its own derivative, and we've already established that $y = e^x$ makes (1.1.1) true. But note that so do $y = 2e^x$, $y = 3e^x$, and more generally, $y = Ce^x$ for any $C \in \mathbb{R}$.

So differential equations (DEs) are already more complicated than algebraic equations in the sense that there can be many solutions to a simple-looking equation. If we want a *unique* solution, generally we have to impose some additional requirements: it may be shown that $y = e^x$ is the *only* solution of the problem

$$\begin{cases} y' = y, \\ y(0) = 1. \end{cases} \tag{1.1.2}$$

Equation (1.1.2) is an example of an *initial value problem* (IVP). But as we will see, not even every IVP has a unique solution, or even a solution at all.

This section introduces the definition of a *differential equation* as well as what it means to *solve* a DE.

Definition 1.1.1 (Differential equation)**.** *An equation involving the derivatives of one or more unknown functions, with respect to one or more independent variables, is said to be a* ***differential equation (DE).***

Classification of DEs

We can classify DEs according to *type, order, and linearity.*

Remark 1.1.1 (Classification by type)**.** "Type" refers to how many independent variables we are taking derivatives of. If we are only taking a single-variable derivative (cf. (1.1.1)), we call the DE an **ordinary differential equation (ODE)**. If partial derivatives appear, then it is a **partial differential equation (PDE)**.

Remark 1.1.2 (Classification by order)**.** "Order" refers to the order of the highest derivative in the DE.

Remark 1.1.3 (Classification by linearity)**.** This is the hardest one for most students. An n-th order ODE is said to be **linear** if it is linear in $y, y', \dots, y^{(n)}$ (e.g., no y'^2, $\sin y$, e^y). *A* ***nonlinear*** *equation is one that is not linear.*

Example 1.1.1. *Classify the following DEs according to their type, order, and linearity:*

(1) $y' + 4y = t$

(2) $(2ty')' + 3y = \sin t$

(3) $\partial_t u = \partial_x^2 u + \partial_y^2 u$

(4) $(\sin^2 y + \frac{1}{2}\cos^2 y)y'' + \frac{1}{2}(\sin 2y)y'^2 = 0.$

Solution.

(1) ODE, first-order, linear.

(2) ODE, second-order, linear.

(3) PDE, second-order, linear.

(4) ODE, second-order, nonlinear.

□

Notice that linear equations can have nonlinear terms in the independent variable (cf. Example 1.1.1). The "linearity" asks for linearity only in the dependent variable (the unknown function and derivatives $y, y', \dots, y^{(n)}$).

We can always write any n-th order ODE in symbols as

$$F(x, y, y', \dots, y^{(n)}) = 0 \tag{1.1.3}$$

for some real-valued continuous function F. Assuming that we can solve (1.1.3) uniquely for the highest-order term $y^{(n)}$, we can rewrite (1.1.3) as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \tag{1.1.4}$$

where f is another real-valued continuous function. For the equations we deal with in this class, you can assume that we will always be able to do this. The form (1.1.4) is called the **normal form** of the ODE (1.1.3).

It will often be convenient for us to write first- and second- order ODE in the normal forms

$$y' = f(x, y)$$

and

$$y'' = f(x, y, y'),$$

respectively.

Example 1.1.2. *Put the following ODEs into normal form:*

(1) $y'' + 5y' - 6y = 5e^x$

(2) $(2ty')' + 3y = \sin t.$

Solution. (1) Rearranging so that y'' is isolated on the LHS, we get

$$y'' = 5e^x + 6y - 5y'.$$

(2) By the product rule,

$$(2ty')' = 2ty'' + 2y'.$$

Thus the ODE becomes

$$2ty'' + 2y' + 3y = \sin t.$$

Notice that this is a second-order linear ODE. Solving this for y'' gives

$$y'' = \frac{1}{2t}\left(\sin t - 3y - 2y'\right), \quad t \neq 0.$$

□

Upshot

- ODEs can be classified according to **type, order**, and **linearity**.
- We usually write ODEs in **normal form**, where the highest-order derivative appears on the LHS and all other terms on the RHS.

First-order ODEs are sometimes also written in the **differential form**

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0. \tag{1.1.5}$$

Assuming that y is the dependent variable and x is the independent variable, recall that we can write the differentials as $\mathrm{d}y = y'\,\mathrm{d}x$ (think about "dividing" $\frac{\mathrm{d}y}{\mathrm{d}x} = y'$ by $\mathrm{d}x$).

Example 1.1.3 (Differential form of ODE)**.**

(1) Put the ODE $y\sin x\,\mathrm{d}x - \mathrm{d}y = 0$ into normal form. Is this equation linear or nonlinear?

(2) Put the ODE $xy\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$ into differential form.

Solution. (1) Dividing the ODE by $\mathrm{d}x$,

$$y\sin x - \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Thus

$$y\sin x - y' = 0 \implies y' = y\sin x.$$

This equation is linear.

(2) Multiplying by $\mathrm{d}x$,

$$xy\,\mathrm{d}y = (x^2 + y^2)\,\mathrm{d}x.$$

Thus the differential form of this ODE is

$$(x^2 + y^2)\,\mathrm{d}x - xy\,\mathrm{d}y = 0.$$

□

Solutions

Definition 1.1.2 (Solution of an ODE). *Given an n-th order ODE*

$$F(x, y, y', \dots, y^{(n)}) = 0,$$

a function u is said to be a ***solution*** *if u is defined on some interval I and*

$$F(x, u(x), u'(x), \dots, u^{(n)}(x)) = 0 \quad (x \in I). \tag{1.1.6}$$

Solutions to an ODE must always come specified with an interval I. The reason is this. For initial value problems like (1.1.2), solutions depend on the boundary condition (e.g., the value of y at 0), so the interval I should contain 0. But if I *only contains* 0 (i.e. it is not an interval), then our solution is really only a solution to an algebraic equation, and not a function.

Definition 1.1.3 (Interval of existence). *The interval I which appears in Definition 1.1.2 is called the* ***interval of existence, interval of definition, or interval of validity****. It is the interval on which the solution to an ODE is defined on.*

Intervals of existence may be open (a, b), closed $[a, b]$, infinite $\mathbb{R}$, or semi-infinite $(a, +\infty)$ or $(-\infty, b)$.

Upshot

Solutions to ODEs always consist of a function satisfying the equation plus its interval of existence.

Example 1.1.4. *Verify that the following are solutions to the given ODE on the interval* $\mathbb{R}$.

(1) $y' = 2y$, $u = e^{2x}$
(2) $y'' - 4y' + 4y = 0$, $u = xe^{2x}$
(3) $y' = \sqrt{y}$, $u = \frac{1}{4}x^2$

Solution. (1) If $u = e^{2x}$, we see

$$u' = 2e^{2x} = 2u,$$

and this holds for all $x \in \mathbb{R}$.

(2) If $u = xe^{2x}$, we calculate

$$u' = e^{2x} + 2xe^{2x}$$

and

$$u'' = 2e^{2x} + 4xe^{2x} + 2e^{2x} = 4e^{2x} + 4xe^{2x}.$$

Hence,

$$\begin{aligned} u'' - 4u' + 4u &= (4e^{2x} + 4xe^{2x}) - 4(e^{2x} + 2xe^{2x}) + 4(xe^{2x}) \\ &= (4e^{2x} - 4e^{2x}) + (4xe^{2x} - 8xe^{2x} + 4xe^{2x}) \\ &= 0 \end{aligned}$$

for all $x \in \mathbb{R}$.

(3) If $u' = \frac{1}{4}x^2$, then

$$u' = \frac{1}{2}x.$$

So

$$u' = \frac{1}{2}x = \sqrt{\frac{1}{4}x^2} = \sqrt{u}.$$

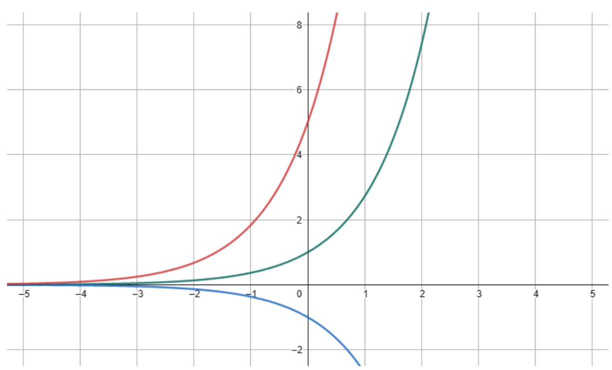
□

Integral curves

Definition 1.1.4 (Integral curve)**.** *The graph of the solution to an ODE (in the (x, y)-plane) is called an* ***integral curve****.*

Be careful here, as the book calls these "solution curves." We will only say "solution curve" when the solution is also the solution to an IVP, that is, it satisfies some boundary/initial conditions. If there are no boundary conditions specified (i.e., we just have the solution to an ODE), we will say "integral curve."

Example 1.1.5. At the beginning of this section, we saw that $u(x) = Ce^x$ for any $C \in \mathbb{R}$ is a solution to the ODE $y' = y$. Thus the graph of any function $u(x) = Ce^x$ is an integral curve. The integral curves $y = 5e^x$, $y = e^x$, and $y = -e^x$ are shown below. The integral curve $y = e^x$ (in green) is the **solution curve** of the IVP (1.1.2).



Upshot

- The graphs of solutions to ODEs are called *integral curves*.
- The graphs of solutions to IVPs (ODE + initial/boundary condition) are called **solution curves**.

Note that, since a solution u to an ODE is differentiable on I, u must also be continuous on I. Thus there might be a disagreement between the domain of the function u and the interval of existence of the solution u. This is highlighted in the next example.

Example 1.1.6 (Function $y = 1/x$ vs solution $y = 1/x$)**.** Recall that the function $y = 1/x$ has domain $(-\infty, 0) \cup (0, +\infty)$ (all real numbers except for 0). At $x = 0$, it has a vertical asymptote, so it is neither continuous nor differentiable there.

On the other hand, $y = 1/x$ is also a solution to the nonlinear first-order ODE $y' = -y^2$ (check this). But again, when we say that this is a *solution* to this ODE, Definition 1.1.2 requires that we specify an *interval* on which it is continuous. Any interval not containing 0 works (e.g., $(1, 2)$, $(-10, -1/2)$, $(-\infty, 0)$ or $(0, +\infty)$). But note that claiming $I = (-\infty, 0) \cup (0, +\infty)$ as the *interval of existence* does not make sense because $(-\infty, 0) \cup (0, +\infty)$ is not an *interval*.

It is customary to take I to be as large as possible: in this case either $(-\infty, 0)$ or $(0, +\infty)$.

Families of solutions

Remember from integral calculus that in taking an antiderivative of some function, we get a $+C$ in our answer. The $+C$ then goes away if we take a *definite* integral. Solutions to ODEs are similar in the sense that in finding a *general* solution to an ODE, we usually develop some parameters C_j. For example, recall that the functions $y(x, C) = Ce^x$ are solutions to the ODE $y' = y$ for any $C \in \mathbb{R}$. As with integration, the C then goes away if we prescribe some additional information (initial/boundary) conditions. For instance, if we also require $y(0) = 1$, then consequently $C = 1$. In this case, we call $y = e^x$ a *particular solution* to the IVP $y' = y$, $y(0) = 1$.

More generally, we will see in Chapter 2 that when solving a first-order ODE $F(x, y, y') = 0$, we usually get a solution containing a single constant/parameter C. We can write a solution of $F(x, y, y') = 0$ in the form $G(x, y, C) = 0$. We call this a **one-parameter family of solutions**. Moreover, when solving an n-th order ODE $F(x, y, y', \ldots, y^{(n)})$, we generally obtain an n**-parameter family of solutions** $G(x, y, C_1, \ldots, C_n) = 0$. This is all to say that a single ODE can possess an infinite number of solutions corresponding to an unlimited number of choices for the parameters (boundary/initial conditions/data). A solution of an ODE which is free of parameters (that comes from the boundary conditions) is called a **particular solution**.

Example 1.1.7.

(1) Verify that the functions $y(x, C_1, C_2) = C_1 e^{2x} + C_2 x e^{2x}$ are a 2-parameter family of solutions to the ODE $y'' - 4y' + 4y = 0$.

(2) Find the particular solution corresponding to the initial conditions $y(0) = 1$, $y'(0) = 0$.

Solution. (1) Letting $y = C_1 e^{2x} + C_2 x e^{2x}$, we calculate

$$y' = 2C_1 e^{2x} + 2C_2 x e^{2x} + C_2 e^{2x}$$

and

$$y'' = 4C_1 e^{2x} + 4C_2 x e^{2x} + 2C_2 e^{2x} + 2C_2 e^{2x} = 4C_1 e^{2x} + 4C_2 e^{2x} + 4C_2 x e^{2x}.$$

So

$$\begin{aligned} y'' - 4y' + 4y &= \\ &= 4C_1 e^{2x} + 4C_2 e^{2x} + 4C_2 x e^{2x} - 4(2C_1 e^{2x} + 2C_2 x e^{2x} + C_2 e^{2x}) + 4(C_1 e^{2x} + C_2 x e^{2x}) \\ &= (4C_1 - 8C_1 + 4C_1) e^{2x} + (4C_2 - 4C_2) e^{2x} + (4C_2 x - 8C_2 x + 4C_2 x) e^{2x} \\ &= 0. \end{aligned}$$

(2) If $y(0) = 1$, then notice that

$$1 = y(0) = C_1 e^0 + 0 = C_1.$$

The condition $y'(0) = 0$ then implies

$$0 = y'(0) = 2C_1 + C_2 = 2 + C_2 \implies C_2 = -2.$$

Thus, the particular solution is
$$y = e^{2x} - 2xe^{2x}.$$
□

Sometimes there are solutions to ODEs which are not part of a family of solutions. For example, the functions $y = \frac{1}{4}(x+C)^2$ are a 1-parameter family of solutions to the equation $y' = \sqrt{y}$ (check this). But $y' = \sqrt{y}$ also has the constant solution $y = 0$, called the **trivial solution**. Note that it is not part of the parameter family, since no value of C makes $\frac{1}{4}(x+C)^2 = 0$ for all x.

Solutions which appear apart from a parameter family are called **singular solutions**.

Systems

Up to this point we have only considered single differential equations containing one unknown function. But in this course and also in applications, we often will deal with systems of differential equations.

Definition 1.1.5 (System)**.** *A **system of ODEs** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.*

For example, if x and y denote the dependent variables and t the independent variable, we can write a system of 2 first-order ODEs in normal form by

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f(t,x,y),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(t,x,y).\end{aligned} \tag{1.1.7}$$

A **solution** to a system (1.1.7) is then a pair of differentiable functions, $x = u_1(t)$ and $y = u_2(t)$, defined on a common interval I, such that

$$\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}t} &= f(t,u_1(t),u_2(t)) \quad (t \in I)\\ \frac{\mathrm{d}u_2}{\mathrm{d}t} &= g(t,u_1(t),u_2(t)) \quad (t \in I).\end{aligned}$$

Summary

- An **ordinary differential equation (ODE)** is an equation involving an unknown function and its derivatives in 1 independent variable.
- Differential equations can be classified according to 3 things: **type, order**, and **linearity**.
- A **solution** to an ODE is a function satisfying the differential equation plus its **interval of existence**.
- An **integral curve** is the graph of the solution of an ODE, and a **solution curve** is the solution of an IVP.

1.2 Initial Value Problems

Learning objectives

1. Distinguish between an ODE and an **initial value problem** (IVP).
2. Know the precise definition of an IVP.
3. Understand how changing the **initial condition** produces changes in the **solution curve** and solution behavior.
4. Given a **general solution** to an ODE, know how to apply initial conditions to find a **particular solution** to an IVP.
5. Understand and be able to apply the **Picard-Lindelöf existence and uniqueness theorem**.
6. Determine **intervals of existence** for solutions to IVPs.

Introduction and motivation

Recall that even first order linear ODEs often have infinitely many solutions (because of the $+C$ that pops up). For example, consider the first-order linear ODE

$$y' = -ky. \tag{1.2.1}$$

Notice that $y = Ce^{-kt}$ is a solution for any $C \in \mathbb{R}$, for

$$y' = -kCe^{-kt} = -ky.$$

As we have already seen, often we also want to specify the constant C that appears. We can do this by specifying a side condition on the boundary, e.g., $y'(0) = y_0$. Then

$$y_0 = y(0) = Ce^{-k(0)} = C,$$

so $y = y_0e^{-kt}$.

Before giving the formal definition of an IVP, consider another example. Carbon-14, a radioactive isotope of carbon, has a half-life of 5730 years. We can model its decay with the ODE

$$y' = -\frac{\ln 2}{5730}y, \tag{1.2.2}$$

with y in units of mass. So according to the discussion above, $y = Ce^{-\frac{\ln 2}{5730}t}$ is a solution for all $C \in \mathbb{R}$. So, if y is in units of "amount," then how do we know which value of C to pick? Well, suppose that we have a 1g sample of Carbon-14. That is, at present-time ($t = 0$), we have $y(0) = 1$g. So this sample satisfies the *initial value problem*

$$\begin{aligned} y' &= -\frac{\ln 2}{5730}y, \\ y(0) &= 1. \end{aligned} \tag{1.2.3}$$

Thus, as we will see later in this section, the unique solution to (1.2.3) is $y = e^{-\frac{\ln 2}{5730}t}$. (This is one of the principles/models used in carbon dating.)

Definition 1.2.1 (Initial value problem (IVP), initial condition)**.** *An **initial value problem (IVP)** is a differential equation*

$$y' = f(t, y)$$

combined with a side condition, where t_0 is a point in the domain of f and $y_0 \in \mathbb{R}$ is given:

$$y(t_0) = y_0.$$

*We call the constraint $y(t_0) = y_0$ the **initial condition**.*

A couple of comments before moving on. First, when you see/hear "IVP," you should think: ODE + initial condition. Second, Definition 1.2.1 is defined in terms of first-order ODE. For IVPs involving higher-order ODE, we need more initial conditions (because more $+C_j$'s pop up).

Definition 1.2.2 (nth-order IVP)**.** *An n**th-order IVP** is an nth-order ODE combined with n initial conditions, that is,*

$$\begin{aligned} y^{(n)} &= f(t, y, y', \ldots, y^{(n-1)}), \\ y(t_0) = y_0, \quad y'(t_0) &= y_1, \quad \ldots, \quad y^{(n-1)}(t_0) = y_{n-1}. \end{aligned} \tag{1.2.4}$$

Upshot

An **initial value problem (IVP)** is an ODE plus initial conditions.

Geometric interpretation

A graph of a solution of an IVP is the graph of the solution of the ODE also passing through a point specified by the initial condition. For example, consider again the ODE (1.2.2). Several integral curves are shown below: Again, each of the curves shown is a solution to the ODE (1.2.2). The

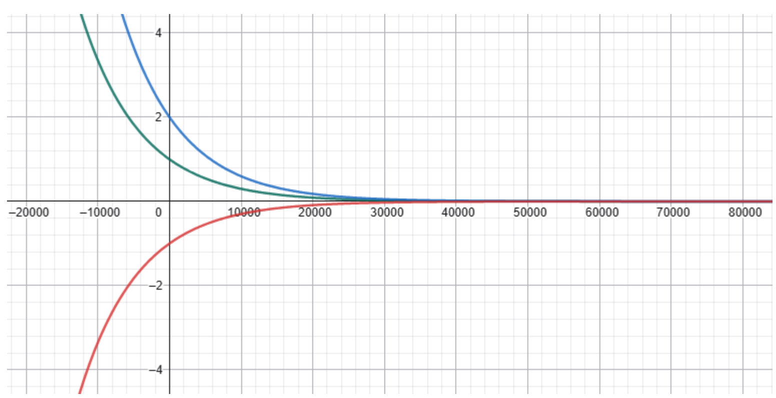


Figure 1.2.1: Integral curves of ODE (1.2.2). Blue: $y(0) = 2$, Green: $y(0) = 1$, Red: $y(0) = -1$.

solution of the IVP (1.2.3) is the integral curve passing through $y = 1$, that is, the one in green. We call this a *solution curve*.

Definition 1.2.3 (Solution curve)**.** *A **solution curve** is the graph of the solution to an IVP.*

Self-Check

What is the difference between an integral curve and a solution curve?

Solution curves of higher-order IVPs work similarly: solution curves are the integral curves passing through the initial conditions. So in this case, the solution must pass through a specified point, but also satisfy some additional requirements on its derivatives:

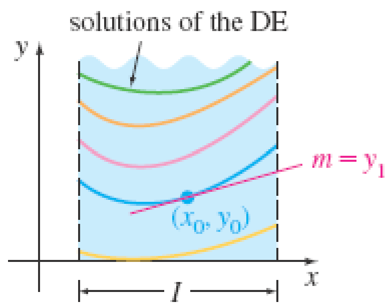


Figure 1.2.2: Integral and solution curves of a second-order ODE/IVP.

Examples

Example 1.2.1. *As discussed at the beginning of this section, $y = Ce^{-kt}$ is a 1-parameter family of solutions for the ODE*

$$y' = -ky.$$

Determine a particular solution for the following choices of initial conditions. Sketch the solution curves on one plot.

1. $y(0) = 1, k = 1$
2. $y(0) = -1, k = 1$
3. $y(0) = 1, k = -1$

Solution. (1) Since $y(0) = 1$, we have

$$1 = y(0) = C.$$

Since $k = 1$, $y(t) = e^{-t}$.
(2) Since $y(0) = -1$, it now follows that $C = -1$, so $y(t) = -e^{-t}$.

(3) Now $C = 1$, but the exponent is different: $y(t) = e^t$.

□

Example 1.2.2. *For each of the following, determine the particular solution and its interval of existence from the general solution and initial conditions. Sketch the solution curve.*

(1) $y' = 2y \sin t, \quad y(t) = Ce^{-2\cos t}, \quad y(\pi/2) = 5.$

(2) $y'' + 5y' - 6y = 0, \quad y(t) = C_1 e^t + C_2 e^{-6t}, \quad y(0) = 1, \quad y'(0) = 0.$

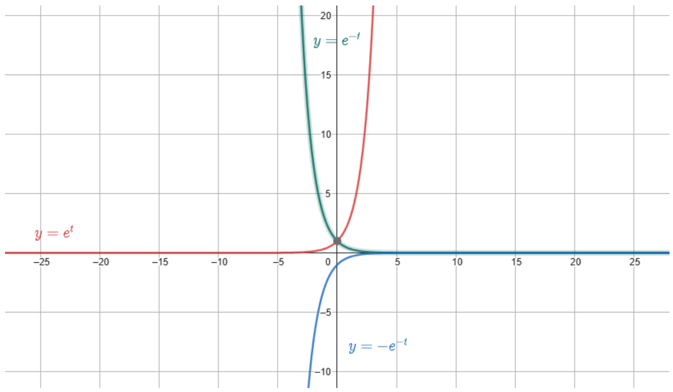


Figure 1.2.3: Solutions of Example 1.2.1. Green: $y = e^{-t}$, Blue: $y = -e^{-t}$, Red: $y = e^t$.

(3) $y' = y^2$, $y(t) = \frac{1}{C-t}$, $y(1) = \frac{1}{4}$.

Solution. (1) Since $\cos \frac{\pi}{2} = 0$,

$$5 = y(\pi/2) = Ce^{-2\cos(\frac{\pi}{2})} = C.$$

Hence $y(t) = 5e^{-2\cos t}, (t \in \mathbb{R})$. This solution has interval of existence $\mathbb{R}$ because it is smooth on $\mathbb{R}$ (composition of smooth functions is smooth).

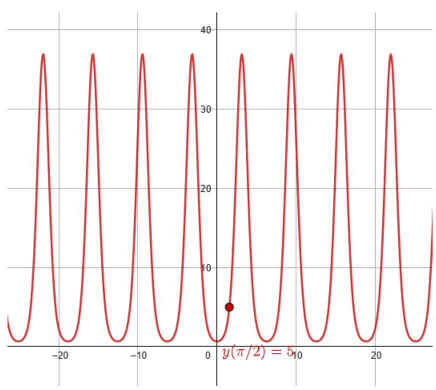


Figure 1.2.4: Solution curve of Example 1.2.2 (1).

(2) We first calculate

$$y'(t) = C_1e^t - 6C_2e^{-6t}.$$

The initial conditions say

$$1 = C_1 + C_2$$
$$0 = C_1 - 6C_2.$$

Thus $C_1 = 6C_2$, so

$$1 = C_1 + C_2 \implies 1 = 6C_2 + C_2 = 7C_2 \implies C_2 = \frac{1}{7} \implies C_1 = \frac{6}{7}.$$

Hence $y(t) = \frac{6}{7}e^t + \frac{1}{7}e^{-6t}, (t \in \mathbb{R})$. This again has interval of existence $\mathbb{R}$ because exponential functions are continuous and twice differentiable on $\mathbb{R}$.

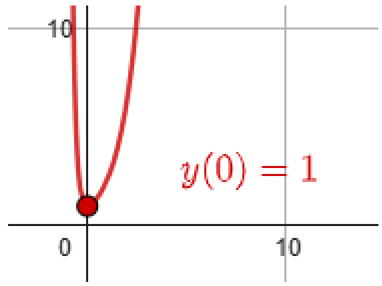


Figure 1.2.5: Solution curve of Example 1.2.2 (2).

(3) We find

$$\frac{1}{4} = y(1) = \frac{1}{C-1} \implies C - 1 = 4 \implies C = 5.$$

Thus $y(t) = \frac{1}{5-t}, (t \in (-\infty, 5))$. Now this solution y has interval of existence $(-\infty, 5)$ because of the discontinuity at $t = 5$. In the case of IVPs, we pick the interval containing the initial condition ($t = 1$ in this case). □

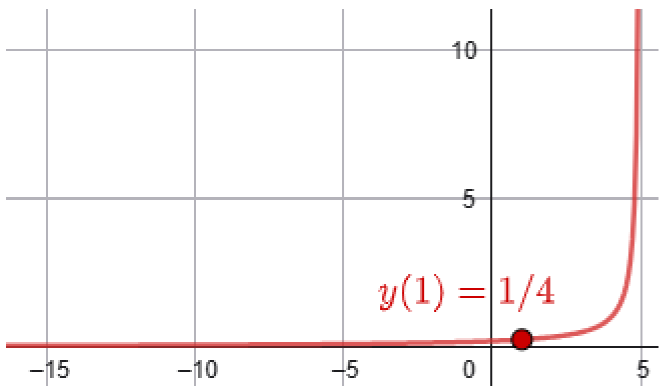


Figure 1.2.6: Solution curve of Example 1.2.2 (3).

Solutions

> **Self-Check**
>
> What does it mean to *solve* an IVP?

The short answer is that a solution to an IVP is a solution to the underlying ODE which also satisfies the initial conditions (think: there are many integral curves, but only 1 passing through a specific point). Since solutions to ODEs must be defined on an interval of existence I, so must solutions to IVPs. Now, since we are specifying a point of interest in the initial condition (e.g. t_0 in $y(t_0) = y_0$), the interval of existence I must now contain t_0. We put the precise definition below.

Definition 1.2.4 (Solution of IVP)**.** *Given an IVP*

$$y' = f(t, y), \quad y(t_0) = y_0,$$

a function u is said to be a ***solution*** *if u is defined on some interval I containing t_0 and*

$$\begin{aligned} u'(t) &= f(t, u(t)) \quad (t \in I), \\ u(t_0) &= y_0. \end{aligned} \tag{1.2.5}$$

Upshot

The **solution** of an IVP is a solution to the underlying **ODE** which also satisfies the **initial conditions**.

Remark 1.2.1 (Terminology: General/particular solution)**.** We will often call the solution to an ODE, that is, parameter families of solutions, a **general solution** to the ODE. For example, $y(t) = Ce^{-2\cos t}$ above is a general solution to the ODE $y' = 2y \sin t$.

We then call the solution to an IVP a **particular solution**. For instance, $y(t) = 5e^{-2\cos t}$ is a particular solution to the IVP

$$y' = 2y \sin t, \quad y(\pi/2) = 5.$$

Existence and uniqueness

We have already seen that many ODEs have infinitely many solutions. What about for IVPs, when we have an initial requirement to satisfy? We first look at 2 examples.

Example 1.2.3 (IVP with no solution)**.** Consider the first-order linear IVP

$$\begin{aligned} ty' &= 2y, \\ y(0) &= 1. \end{aligned} \tag{1.2.6}$$

The ODE $ty' = 2y$ has as its general solution

$$y(x) = Ct^2 \quad (t \in (-\infty, 0) \text{ or } t \in (0, +\infty)).$$

Note that

$$ty' = t(2Ct) = 2Ct^2 = 2y$$

holds for all $t \in \mathbb{R}$, but the interval of existence comes from when we put the ODE into normal form:

$$y' = 2\frac{y}{t}.$$

Here $f(t, y) = 2\frac{y}{t}$ is continuous for all $y \in \mathbb{R}$ but only for t away from 0. The takeaway is this. Since the initial condition specifies $y(0) = 1$, the initial condition asks that the solution y be *defined* at $t = 0$. But what it means for y to be a solution to the IVP at $t = 0$ is that it also solves the ODE

$$y' = 2\frac{y}{t}$$

on some interval I containing $t = 0$ (cf. Definition 1.2.4) But this is impossible since division by 0 is impossible. Hence, there is no solution to IVP (1.2.6).

Example 1.2.4 (IVP with multiple solutions)**.** Consider the first-order nonlinear IVP

$$\begin{aligned} y' &= 2\sqrt{y}, \\ y(0) &= 0. \end{aligned} \tag{1.2.7}$$

The ODE $y' = 2\sqrt{y}$ has general solution

$$y(t) = (t + C)^2.$$

So from the initial condition $y(0) = 0$, we determine

$$0 = y(0) = C^2 \implies C = 0.$$

Thus $y(t) = t^2$ is a solution to IVP (1.2.7). But observe that so is the constant function $y \equiv 0$. So we see that the IVP (1.2.7) has multiple solutions.

So from the above examples we see that there is no guarantee that an IVP will even have a solution, or that a solution will be unique, if it exists.

Upshot

An arbitrary IVP may have no solutions, 1 unique solution, or (infinitely) many.

Hopefully it is clear why we would like to ensure that a solution to a given IVP exists, or else you wouldn't be in a course where you learn how to solve them. On the other hand, we often also want to ensure that solutions to IVPs are *unique* as well.

Self-Check

Consider if IVP (1.2.3) had multiple solutions. What would this mean physically? How would carbon dating work?

The following theorem establishes conditions under which an initial value problem has a unique solution. It is named after Émile Picard (1856–1941) and Ernst Lindelöf (1870–1946).

Theorem 1.2.1 (Picard–Lindelöf existence and uniqueness theorem)**.** *Consider the initial value problem*

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{1.2.8}$$

Let $D = [a, b] \times [c, d]$ be a closed rectangle containing the point (t_0, y_0) in its interior, that is, D is such that

$$a \le x \le b, \quad c \le y \le d, \quad a < t_0 < b, \quad c < y_0 < d.$$

If f and $\frac{\partial f}{\partial y}$ are continuous on D, then there exists some interval $I = (t_0 - h, t_0 + h) \subset [a, b]$ containing t_0, and a function $u(t)$, defined on I, such that u is the unique solution to IVP (1.2.8) on I.

The requirement that f is continuous means that solutions to IVPs can't jump: if $y' = f(t, y)$ and f is discontinuous, then y' at least has sharp kinks/corners, so y is not differentiable.

The requirement that $\partial_y f$ is continuous means that solutions must take a specific "trajectory" through the initial condition $y(t_0) = y_0$. If $\partial_y f$ is *not* continuous, then it can happen that we can find multiple solutions y passing through (t_0, y_0) but with different values of $y''(t_0)$ (see Example 1.2.4). This produces different solutions, since solutions can "split apart" after passing through the initial condition.

In fact, it is possible to say a little bit more about existence/uniqueness.

Remark 1.2.2 (Existence/uniqueness)**.** Consider the IVP

$$y' = f(t, y), \quad y(t_0) = y_0.$$

(1) If $f(t, y)$ is not continuous at $f(t_0, y_0)$, then there may not be a solution to the IVP.

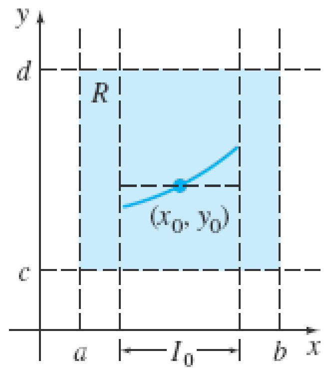


Figure 1.2.7: Visualization of Theorem 1.2.1

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(2) If $f(t,y)$ is continuous at $f(t_0,y_0)$ but $\partial_y f(t,y)$ is not, then there exists a solution to the IVP (but may not be unique).

Example 1.2.5. *What condition of Theorem 1.2.1 fails in Example 1.2.3? In Example 1.2.4?*

Solution. In Example 1.2.3, we have the ODE

$$ty' = 2y,$$

which has normal form

$$y' = 2\frac{y}{t}.$$

So here $f(t,y) = 2\frac{y}{t}$, so f is not continuous at $t = 0$. This is exactly where the initial condition is, so there is no solution.

In Example 1.2.4, we are given the ODE in normal form:

$$y' = 2\sqrt{y}.$$

So here $f(t,y) = 2\sqrt{y}$, so f is continuous for all $(t,y) \in \mathbb{R}^2$. However, differentiating f, we find

$$\partial_y f = \frac{1}{\sqrt{y}},$$

so $\partial_y f$ is not continuous at $y = 0$. Since the initial condition is $y(0) = 0$, this means that solutions are not (necessarily) unique. $\square$

Example 1.2.6. *Use Theorem 1.2.1 to determine if there exists a unique solution to the following IVPs.*

(1) $y' = -2\sqrt{xy}, \quad y(x_0) = 0$

(2) $y' = -2\sqrt{xy}, \quad y(x_0) = 1$

(3) $y' = \frac{y}{t^2}, \quad y(0) = 0$

(4) $y' = \frac{y}{t^2}, \quad y(-1) = 0$

Solution. (1) Here $f(x,y) = -2\sqrt{xy}$, which is continuous for all $(x,y) \in \mathbb{R}^2$. We compute

$$\partial_y f(x,y) = -2\sqrt{x} \cdot \frac{1}{2\sqrt{y}} = -\sqrt{\frac{x}{y}}.$$

Thus $\partial_y f(x,y)$ is not continuous at $y = 0$, so there is not a unique solution to this IVP.

(2) In this case, we still have $\partial_y f(x,y) = -\sqrt{x/y}$, but now the initial condition is $y(x_0) = 1$. Since $\partial_y f(x,y)$ is continuous on $\mathbb{R} \times (0, +\infty)$, which contains the initial condition $(x_0, 1)$, Theorem 1.2.1 implies that there exists a unique solution to the IVP.

(3) We have $f(t,y) = y/t^2$, which is not continuous at $t = 0$. Thus the assumptions of Theorem 1.2.1 are not satisfied, so there may not be a solution to the IVP.

(4) We still have $f(t,y) = y/t^2$ but now with initial condition $(-1, 0)$. Note that $f(t,y)$ is continuous on $(-\infty, 0) \times \mathbb{R}$, which contains $(-1, 0)$. We calculate

$$\partial_y f(t,y) = \frac{1}{t^2},$$

which again is continuous on $(-\infty, 0) \times \mathbb{R}$. Hence, Theorem 1.2.1 guarantees the existence of a unique solution to the IVP. □

Summary

- An **initial value problem (IVP)** consists of an ODE plus **initial conditions**.
- Graphs of solutions to IVPs are called **solution curves**.
- Know how to apply initial conditions to a **general solution** of an ODE to obtain a **particular solution** to an IVP.
- IVPs may or may not have solutions, and if they do, they may not be unique. The Picard-Lindelöf existence and uniqueness theorem (cf. Theorem 1.2.1) gives us a way to determine if an IVP has a unique solution or not.

1.3 Differential Equations as mathematical models

Learning objectives

1. Understand why ODEs are used in **mathematical models** in many applications.
2. Explain how models arise from **physical laws** about the **proportionality** of a quantity and its rates.
3. Describe what the **solution** to an ODE/IVP means in the context of a modeling problem.
4. Introduce several models involving ODEs and discuss their derivation.

Introduction and motivation

In many applications it is assumed that some quantity is related to its rate (population depends on growth rate, force applied depends on distance from equilibrium, cooling rate depends on temperature difference between hot thing and room temperature). Since derivatives are rates of change, in these situations we frequently model these systems or phenomena with **mathematical models** involving ODEs. (You will investigate one such model (diffusion) in depth in Project 1.)

A solution then tells us the **state** of the system which the model describes. That is, knowing the solution means knowing the values of the dependent variable at times t in the past, present, and future in reference to the initial condition (since our solution is defined on an interval). For example, in the carbon dating example (cf. IVP (1.2.3)), the solution y tells us the amount of Carbon-14 remaining, in g, of the original 1g sample after time t, in years.

Upshot

Models of many physical and social phenomena involve ODEs because many quantities depend on their rates.

In later sections we will develop methods for how to solve ODEs and IVPs. In this section, we present models involving ODEs used in other applications. For now, just try to understand why these equations make sense as models.

Proportionality

As we will see, in many models it is assumed that some quantity is *proportional to* its rate of change. We briefly review what this means.

Definition 1.3.1 (Direct proportionality)**.** *We say that A* ***is (directly) proportional to*** *B if*

$$A = kB$$

for some constant k. We write $A \propto B$.

Definition 1.3.2 (Inverse proportionality)**.** *We say that A* ***is inversely proportional to*** *B if*

$$A = \frac{k}{B}$$

for some constant k. We write $A \propto \frac{1}{B}$.

Population dynamics

The idea behind one of the earliest population models is that the growth rate is proportional to the total population at that time. That is, the more people there are at time t, the more people there will be in the future. Let the population be P. In symbols, this says

$$\frac{\mathrm{d}P}{\mathrm{d}t} \propto P.$$

If we also suppose that that there are P_0 people in the population today, we have $P(0) = P_0$. Then we can model this situation with the IVP

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &= kP, \\ P(0) &= P_0. \end{aligned} \tag{1.3.1}$$

Here k is a constant of proportionality ($k > 0$ for growth). Of course this is a simple model; it doesn't take into account for example deaths, sporadic events (COVID), or immigration/emigration.

Decay

The nucleus of an atom or isotope consists of several protons and neutrons surrounded by various electron shells/orbitals. Many isotopes are radioactive/unstable, meaning that they spontaneously emit protons/neutrons from the nucleus.

For example, Carbon-14, which we considered in Section 1.2, has a nucleus consisting of 6 protons and 8 neutrons. The most stable isotope of carbon is Carbon-12, which has 6 protons and 6 neutrons. Thus, over time, Carbon-14 will decay to become Carbon-12.

The law underlying the model for decay is the same as that of population growth: that the rate at which a substance decays is proportional to how much of it is left:

$$\frac{\mathrm{d}A}{\mathrm{d}t} \propto A.$$

So if we have $A(0) = A_0$ grams of substance initially at time $t = 0$, we can model decay with the IVP

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= kA, \\ A(0) &= A_0, \end{aligned} \tag{1.3.2}$$

where k is a constant (here $k < 0$ for decay).

Notice that the ODEs appearing in the IVPs (1.3.1) and (1.3.2) are exactly the same. The only difference is that for growth, we have $k > 0$, and for decay, $k < 0$. Notice that the ODE $y' = ky$ has the general solution $y(t) = Ce^{ky}$, so that if $k > 0$, solutions are exponentially increasing, and if $k < 0$, solutions are exponentially decreasing (cf. Figure 1.3.1).

Upshot

Population growth and **decay** are both modeled with the IVP

$$\begin{aligned} y' &= ky, \\ y(0) &= y_0. \end{aligned}$$

If $k > 0$, the ODE models growth, and if $k < 0$, the ODE models decay.

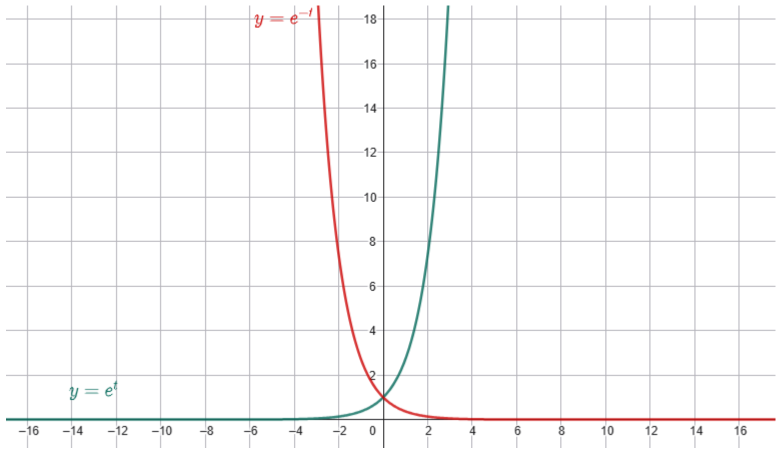

Figure 1.3.1: Solutions to $y' = ky$.

Newton's law of cooling

Think about how a hot dish of food cools down when you take it out of the oven: it cools down quickly at first (oven temp to edible in $\sim$ minutes), but it can take hours for it to cool thoroughly in the refrigerator ($\sim$ hours).

This is because the rate of cooling of a substance is driven by the difference between that substance's current temperature and the temperature of its surroundings. If we let T be the current temperature and T_m the ambient temperature, **Newton's law of cooling** says

$$\frac{\mathrm{d}T}{\mathrm{d}t} \propto (T - T_m).$$

So if a substance has initial temperature $T(0) = T_0$, we can model its temperature profile with the IVP

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}t} &= k(T - T_m), \\ T(0) &= T_0. \end{aligned} \tag{1.3.3}$$

Here $k < 0$ is a constant.

Self-Check

Why is $k < 0$ in IVP (1.3.3)? What would the law predict if $k > 0$?

Example 1.3.1. Suppose that you pour a cup of coffee at a temperature of 80°C in a room with an ambient temperature of 20°C.

(1) Write a first-order IVP describing this situation. (It will include a constant k.) Assume that t is in minutes.

(2) Show that $T(t) = 20 + 60e^{kt}$ is a solution to the IVP you wrote in (1). What is its interval of existence?

(3) If $T(5) = 65$, find k. What units does k have? What does $T(5) = 5$ mean physically?

Solution. (1) Here $T_m = 20$ and $T(0) = 80$ because the coffee has an initial temp of 80°C. We can model this with Newton's law of cooling:

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}t} &= k(T-20), \\ T(0) &= 80. \end{aligned}$$

(2) Differentiating,

$$T'(t) = 60ke^{kt}.$$

So

$$k(T-20) = k\big((20+60e^{kt})-20\big) = 60ke^{kt} = T'(t).$$

Also, we have $T(0) = 20 + 60e^0 = 80$, so $T(t)$ is a solution. Since exponential functions are continuous and differentiable on $\mathbb{R}$, T has interval of existence $\mathbb{R}$.

(3) Notice

$$65 = T(5) = 20 + 60e^{5k} \implies e^{5k} = \frac{45}{60} \implies 5k = \ln\frac{3}{4} \implies k = \frac{1}{4}\ln\frac{3}{4} \approx -0.057.$$

Since T has units in °C, $\frac{\mathrm{d}T}{\mathrm{d}t}$ has units in °C/min. For the ODE in the IVP in (1) to make sense dimensionally, k has units in 1/min. The condition $T(5) = 65$ means that after 5 minutes, the coffee has cooled to 65°C. □

Disease spread

Contagious diseases spread through communities by infected people coming into contact with other (noninfected) people. Let $I(t)$ be the number of infected people at time t, and $S(t)$ the number of noninfected (susceptible) people at time t. The idea here is that the rate at which the infection spreads $\frac{\mathrm{d}I}{\mathrm{d}t}$ is proportional to the number of interactions between these two groups. That is,

$$\frac{\mathrm{d}I}{\mathrm{d}t} \propto SI,$$

or $\frac{\mathrm{d}I}{\mathrm{d}t}$ is *jointly proportional* to $S(t)$ and $I(t)$.

Now let the total number of people in a community be N. If 1 infected person is introduced to this community, then S and I are related by

$$S(t) + I(t) = N + 1.$$

Using this in the proportionality, we get

$$\frac{\mathrm{d}I}{\mathrm{d}t} \propto I(N+1-I).$$

Thus we can model infectious disease spread with the following IVP:

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} &= kI(N+1-I), \\ I(0) &= 1, \end{aligned} \tag{1.3.4}$$

where k is a constant of proportionality.

Self-Check

Why is $I(0) = 1$ a reasonable initial condition in IVP (1.3.4)?

Example 1.3.2.

(1) Classify the ODE appearing in IVP (1.3.4).

(2) Does IVP (1.3.4) have a unique solution for any k, N?

Solution. (1) This is a first-order nonlinear ODE. It is nonlinear because of the I^2 term if we distribute:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = k\big[(N+1)I - I^2\big].$$

(2) Let $f(t, I) = kI(N+1-I)$, and notice that f is continuous for all $(t, I) \in \mathbb{R}^2$. Differentiating,

$$\partial_I f(t, I) = k(N+1-I) - kI = k(N+1) - 2kI.$$

Thus $\partial_I f(t, I)$ is also continuous on $\mathbb{R}^2$. Hence, Theorem 1.2.1 implies that there exists a unique solution to IVP (1.3.4). □

Mixtures

Suppose that a large tank holds a volume of V_0 liters (L), filled with some initial concentration C_0 of, say, brine (salt + water), in g/L. Another brine solution with concentration C_{in} is pumped into the tank at a constant rate r, in L/min. As the brine solution is stirred, it is pumped out of the tank at the same rate r. See Figure 1.3.2

Let $A(t)$ denote the amount of salt in the tank, in g, at time t, in min. The rate at which the amount of salt changes is modeled according to

$$\frac{\mathrm{d}A}{\mathrm{d}t} = R_{\text{in}} - R_{\text{out}}, \tag{1.3.5}$$

where R_{in} denotes the rate that salt enters the tank and R_{out} denotes the rate at which salt leaves the tank.

To find R_{in}, we just take the product of the concentration of the solution entering the tank with the rate at which it enters:

$$R_{\text{in}} = C_{\text{in}} \,\frac{\text{g}}{\text{L}} \cdot r \,\frac{\text{L}}{\text{min}} = C_{\text{in}} r \,\frac{\text{g}}{\text{min}}. \tag{1.3.6}$$

To find R_{out}, we do the same. But what is the concentration of the solution leaving the tank? It's just the current amount $A(t)$ divided by the volume V_0. So we get

$$R_{\text{out}} = \left(\frac{A(t)\,\text{g}}{V_0\,\text{L}}\right) \cdot r \,\frac{\text{L}}{\text{min}} = \frac{r}{V_0} A(t) \,\frac{\text{g}}{\text{min}}. \tag{1.3.7}$$

Combining (1.3.5), (1.3.6), and (1.3.7) gives

$$\frac{\mathrm{d}A}{\mathrm{d}t} = C_{\text{in}} r - \frac{r}{V_0} A(t).$$

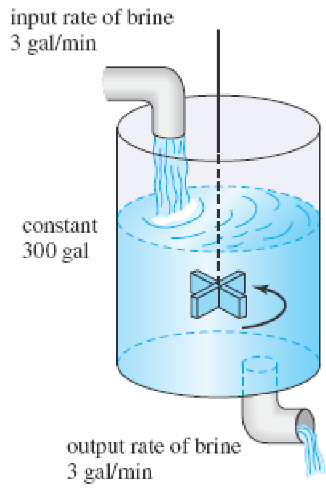

Figure 1.3.2: A tank with $V_0 = 300$ gal, $r = 3$ gal/min.

Thus, if the tank has initial concentration C_0 and volume V_0, then $A(t)$ can be modeled with the IVP

$$\begin{aligned} \frac{dA}{dt} &= C_{in} r - \frac{r}{V_0} A(t), \\ A(0) &= C_0 V_0. \end{aligned} \tag{1.3.8}$$

Here C_0 denotes the initial concentration of the tank, C_{in} denotes the concentration of the brine flowing into the tank, r denotes the rate at which brine enters/exits the tank, and V_0 denotes the volume of brine in the tank.

Example 1.3.3. Brine containing a concentration of 300 g/L salt is pumped into a tank containing 1000 L of brine with an initial concentration of 1 g/L salt at a constant rate of 3 L/min. As the tank is stirred, brine is pumped out at the same rate.

(1) Write an initial value problem describing this situation.

(2) Suppose that now brine is pumped out of the tank at a constant rate of 4 L/min. Write a new IVP describing this situation.

Solution. (1) Here $C_0 = 1$, $C_{in} = 300$, $r = 3$, and $V_0 = 1000$. Thus

$$\begin{aligned} \frac{dA}{dt} &= 900 - \frac{3}{1000} A(t), \\ A(0) &= 1000. \end{aligned}$$

(2) We still have

$$\frac{dA}{dt} = R_{in} - R_{out},$$

but now R_{out} changes. The rate flowing out is now $r_{out} = 4$ L/min, but notice that the volume of the brine in the tank also changes (since $r_{in} = 3 < 4 = r_{out}$, the tank is draining):

$$V(t) = V_0 - \frac{r_{in}}{out} t = 1000 - \frac{3}{4} t.$$

Inserting into (1.3.7),

$$R_{\text{out}} = \frac{A(t)}{V(t)} \cdot r_{\text{out}} = \frac{4A(t)}{1000 - \frac{3}{4}t}.$$

Hence, we get the new IVP

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= 900 - \frac{4A(t)}{1000 - \frac{3}{4}t}, \\ A(0) &= 1000. \end{aligned}$$

Note that the initial condition does not change because the volume of brine in the tank at time $t = 0$ is still the same, $V_0 = 1000$ L. □

Upshot

- If $r_{\text{in}} < r_{\text{out}}$: the number of gallons of brine in the tank is **decreasing**.
- If $r_{\text{in}} > r_{\text{out}}$: the number of gallons of brine in the tank is **increasing**.

In either case, the **volume** of brine in the tank changes with t.

Falling bodies

So far all the models we've introduced have been first-order. Consider now the problem of a falling object with mass m. According to Newton's Second Law of Motion, the force acting on this object in its descent is proportional to its acceleration, $F \propto a$. More precisely, Newton's Second Law states

$$F = ma.$$

On one hand, we have

$$F = -mg$$

, because the force of gravity is constantly applied to the object of mass m. The sign is negative because gravity acts downward (relative to the falling object).

On the other hand, let $s(t)$ denote the position of the object relative to the ground at time t. Then its acceleration a is equal to

$$a = s''(t).$$

Putting the above together, we get

$$-mg = F = ma = ms''(t) \implies s''(t) = -g.$$

Thus, we can model falling objects with the second-order IVP

$$\begin{aligned} \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} &= -g, \\ s(0) &= s_0, \quad s'(0) = v_0, \end{aligned} \tag{1.3.9}$$

where $s(0) = s_0$ denotes the initial position and $s'(0) = v_0$ the initial velocity of the object.

Self-Check

Why are there two initial conditions in IVP (1.3.9)?

Example 1.3.4. Find the solution to IVP (1.3.9).

Solution. Note that the RHS is just a constant, so we can just integrate both sides:

$$s'(t) = \int s''(t)\,\mathrm{d}t = \int -g\,\mathrm{d}t = -gt + C_1.$$

We can integrate again, and we get

$$s(t) = \int s'(t)\,\mathrm{d}t = \int -gt + C_1\,\mathrm{d}t = -\frac{g}{2}t^2 + C_1 t + C_2.$$

Since $s(0) = s_0$,

$$s_0 = s(0) = C_2.$$

Similarly, since $s'(0) = v_0$ and $s'(t) = -2gt + C_1$, we get

$$v_0 = s'(0) = C_1.$$

Hence, $s(t) = -\frac{g}{2}t^2 + v_0 t + s_0$, $t \in \mathbb{R}$. □

Other models and conclusion

We have already considered a handful of models of physical behaviors which involve ODEs, but there are countless others. In §1.3, Zill discusses more applications in chemical reactions, tank draining, and series circuits. As mentioned earlier, you will examine Fick's first law, which models diffusion, in Project 1.

The field of ODEs is still an active area of research in mathematics, partly because of newer, more complex, and more physically-relevant models for physical phenomena are always being developed. For example, in my dissertation, I looked at the following model for the free energy of ferroelectric smectic-A liquid crystal:

$$\begin{aligned}\left(\sin^2\theta + \kappa\cos^2\theta\right)\theta'' + \frac{1}{2}(1-\kappa)\sin(2\theta)(\theta')^2 + \frac{1}{2}\left(\frac{L}{\xi}\right)^2\left(2\widetilde{E}_B\cos\theta - \cos^2\theta\right) &= 0,\\ \left[-\widetilde{W}_S\frac{L}{\xi}\sin\theta + \left(\sin^2\theta + \kappa\cos^2\theta\right)\theta'\right]\Bigg|_{x=0,1} &= 0.\end{aligned}$$

Among other things studied, one of the first questions we had to answer was whether this problem has a solution at all, and we were only able to conclude this under specific conditions.

So the point is this: ODEs are used ubiquitously throughout the sciences to model all kinds of behaviors/phenomena, as these models are usually the result of some physical law about how some quantity is related to one of its rates. Countless existing models are used everyday in numerous industrial settings, and new models are constantly being developed and analyzed in the name of scientific/technological research and development. To this end, knowing how to say something about some of these models allows you to participate in these efforts (or at least understand them to a greater degree than most people in the workforce). Thus, **learning how to explain/interpret a model involving ODEs** is one of the most important skills you will learn in this course.

We now have most of the introductory/background information we need for the rest of this course. In Chapter 2, we begin to discuss solution methods for first-order IVPs.

Summary

- Many **physical laws** establish a **proportionality** between some quantity and its rates. Consequently ODEs are used in **mathematical models** throughout the sciences because they establish a mathematical relation between a dependent variable and its rates (derivatives).
- In the context of a modeling problem, **solutions to IVPs** represent the **“state"** of a physical system at any time t in its interval of existence.
- Several models involving ODEs/IVPs were derived: **population growth, decay, cooling, susceptible/infected, tank problems, and falling bodies**.

Chapter 2

First-order differential equations

2.1 Solution curves without a solution

Learning objectives

1. Interpret the **normal form** of an ODE as a rule for the **slope** of its solutions.
2. Sketch and interpret the **slope field** of a first-order ODE.
3. Use slope fields to sketch **integral curves** and **solution curves** of solutions to ODEs/IVPs.
4. Define what it means for an ODE to be **autonomous**.
5. Know what the slope fields of autonomous ODEs look like.
6. Determine **equilibrium solutions** and interpret **phase portraits** of autonomous ODEs. Classify these equilibrium solutions as **stable, unstable**, or **semi-stable**.

Introduction

Consider the following first-order IVP in normal form:

$$\begin{aligned} y' &= f(t,y), \\ y(t_0) &= y_0. \end{aligned} \tag{2.1.1}$$

Also imagine that we do not know the solution to this IVP and that we cannot think of a method to find one.

It turns out that we can still say something about what solutions to IVP (2.1.1) might "look" like, even if we can't find an explicit, closed-form solution. As we will see, one of the main benefits of putting ODEs into normal form is that they give us a clean way to see how solutions behave or how solution curves evolve.

Upshot

- It is usually possible to analyze how solutions to IVPs behave, even if we can't find a solution explicitly.
- Normal form helps us see what integral curves "look" like.

For example, we can plug in different values of (t,y) to f in order to find the slope y' of any integral curve of $y'(t) = f(t,y)$. The solution to IVP (2.1.1) is then the one passing through the point (t_0, y_0). This idea is explored in this section.

Slope

Recall from Definition 1.2.4 that the solution to the IVP

$$y' = f(t,y), \quad y'(t_0) = y_0$$

is necessarily a differentiable function u on some interval of existence I containing t_0. Thus, u has a tangent line defined at all points t in I.

Also recall that the derivative $u'(t)$ gives the slope of the tangent line to u for all t in I. But if u solves the ODE

$$y'(t) = f(t,y)$$

on I, then this literally gives us a rule for what $u'(t)$ has to be on I.

Definition 2.1.1 (Rate function)**.** *The function $f(t, y)$ in IVP (2.1.1) is called the* ***rate function, slope function, or forcing function****.*

> **Upshot**
>
> If u solves the IVP (2.1.1) on I, then
>
> $$u'(t) = f(t, u(t)) \quad (t \in I).$$

Example 2.1.1. Consider the IVP

$$\begin{aligned} y' &= ty, \\ y(1) &= 2. \end{aligned} \tag{2.1.2}$$

(1) Show that $y' = ty$ has general solution $y = Ce^{t^2/2}$, $t \in \mathbb{R}$.

(2) Find the particular solution to the IVP.

(3) Using your solution in (2), find $y'(0)$, $y'(1)$.

(4) Using the IVP (2.1.2)/rate function $f(t, y) = ty$, find $y'(0)$, $y'(1)$. Compare with (3).

Solution. (1) If $y = Ce^{t^2/2}$, then

$$y' = C \cdot \frac{2t}{2} \cdot e^{\frac{t^2}{2}} = Cte^{\frac{t^2}{2}} = ty, \quad t \in \mathbb{R}.$$

The interval of existence is because $e^{t^2/2}$ is smooth.

(2) Since $y(1) = 2$,

$$2 = y(1) = Ce^{\frac{(1)^2}{2}} = Ce^{\frac{1}{2}} \implies C = \frac{2}{e^{\frac{1}{2}}} = 2e^{\frac{-1}{2}}.$$

Hence,

$$y(t) = \left(2e^{\frac{-1}{2}}\right)e^{\frac{t^2}{2}}.$$

(3) Differentiating,

$$y'(t) = \left(2e^{\frac{-1}{2}}\right) \cdot \frac{2t}{2} e^{\frac{t^2}{2}} = 2te^{\frac{-1}{2}} \cdot e^{\frac{t^2}{2}}.$$

Thus

$$\begin{aligned} y'(0) &= 2(0)e^{\frac{-1}{2}} \cdot 1 = 0, \\ y'(1) &= 2(1)e^{-\frac{1}{2}} \cdot e^{\frac{(1)^2}{2}} = 2e^{\frac{-1}{2}+\frac{1}{2}} = 2. \end{aligned}$$

(4) We first see that

$$y'(0) = (0)y(0) = 0.$$

Next, we apply the initial condition in IVP (2.1.2) to find

$$y'(1) = (1)y(1) = (1)(2) = 2.$$

These match the values found in (3). □

Slope/direction fields

We saw how we can use the RHS of a first-order ODE to find the slope of its solution. If we do this on a rectangular grid and plot the slopes at each point, we get what is called a **slope field** for the ODE.

Definition 2.1.2 (Slope field)**.** *Given an IVP*

$$y' = f(t, y), \quad y(t_0) = y_0,$$

*a **slope field** or **direction field** is a plot, containing the point (t_0, y_0), of line segments, each located at (t, y), with slope $f(t, y)$.*

Slope fields are generally best learned/understood through examples.

Example 2.1.2.

(1) Sketch the slope field for the ODE appearing in the previous example, $y' = ty$.

(2) Using your slope field from (1), sketch solution curves corresponding to the following initial conditions: $y(1) = 2$, $y(2) = 1$, $y(1) = -2$. What happens as $t \to +\infty$ in each of these cases?

Solution. (1) We have already seen how to use the rate function to calculate slope, so we only briefly explain some of the calculations here. When $t = 0$ or $y = 0$, notice

$$y' = f(t, y) = ty = 0.$$

so the slopes are horizontal on each of the two axes (in the slope field, this appears as horizontal dashes on the axes). When $t = 1$, we have

$$y' = f(t, y) = ty = y,$$

so the slopes increase as y increases ($y' \propto y$). Similarly when $y = 1$,

$$y' = f(t, y) = ty = t,$$

so slopes increase as t increases as well.

(2) For the solution curves corresponding to $y(1) = 2$, $y(2) = 1$, we have $y \to +\infty$ as $t \to +\infty$. For the solution curve corresponding to $y(1) = -2$, we have $y \to -\infty$ as $t \to +\infty$. □

Self-Check

What would the solution curve look like for the initial condition $y(1) = 0$?

Example 2.1.3. Consider the ODE

$$y' = y \sin t \tag{2.1.3}$$

(1) Sketch a slope field for the ODE.

(2) Sketch solution curves for the initial conditions $y(0) = 1$ and $y(0) = -2$. What happens as $t \to +\infty$ in this case?

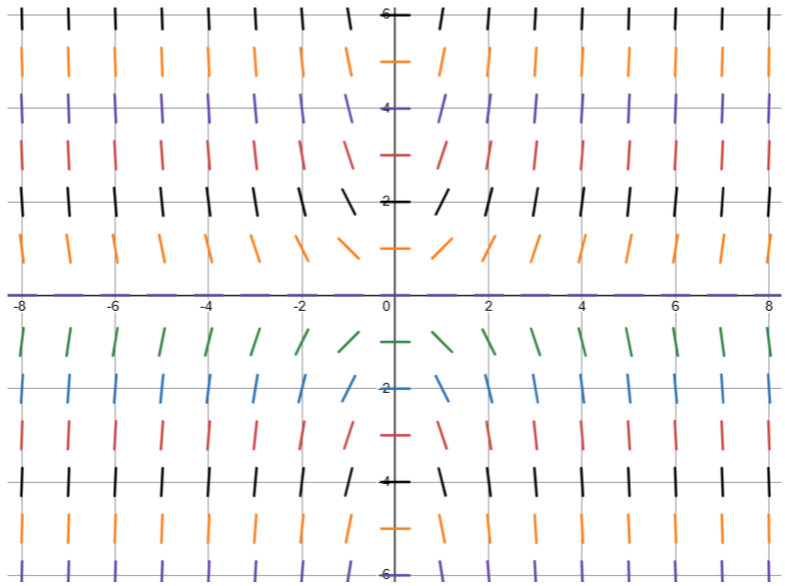

Figure 2.1.1: Slope field of $y' = ty$.

Solution. (1) We again only highlight some of the features in this slope field. Note:

$$y' = y \sin t = 0 \quad (y = 0 \text{ or } t = \pi k, k \in \mathbb{Z}),$$

$$y' = y \sin t = y, \quad (t = \pi/2 + 2\pi k, k \in \mathbb{Z}),$$

and

$$y' = y \sin t = -y, \quad (t = -\pi/2 + 2\pi k, k \in Z).$$

(2) In each case, we get oscillations between $y = 1$ or $y = -2$ and some other value infinitely often. So in both cases, the solutions don't approach a limit as $t \to +\infty$ (think about taking the limit of $\sin t$ or $\cos t$ as $t \to +\infty$). Thus

$$\lim_{to \to +\infty} y(t) = DNE.$$

□

Upshot

Slope fields tell us what solutions look like and how they might behave, even if we can't find one explicitly.

Autonomous first-order ODEs

In Chapter 1 we classified differential equations according to their type (ordinary/partial), order (how many derivatives?) and linearity (linear or nonlinear). Here we introduce another kind of classification. Many ODEs do not involve the independent variable. For first order, these equations look like

$$y' = f(y) \tag{2.1.4}$$

(instead of $f(t, y)$). These equations are said to be **autonomous**.

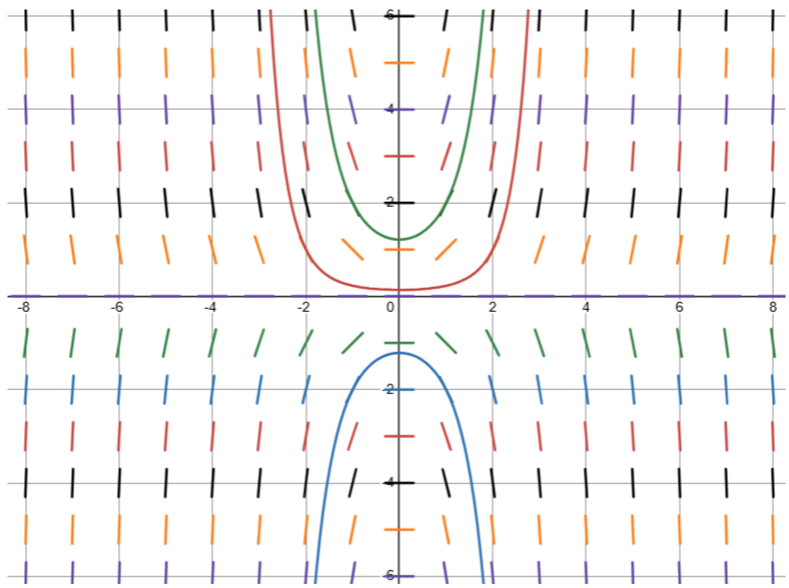

Figure 2.1.2: Solution curves of $y' = ty$. Green: $y(1) = 2$, Red: $y(2) = 1$, Blue: $y(1) = -2$.

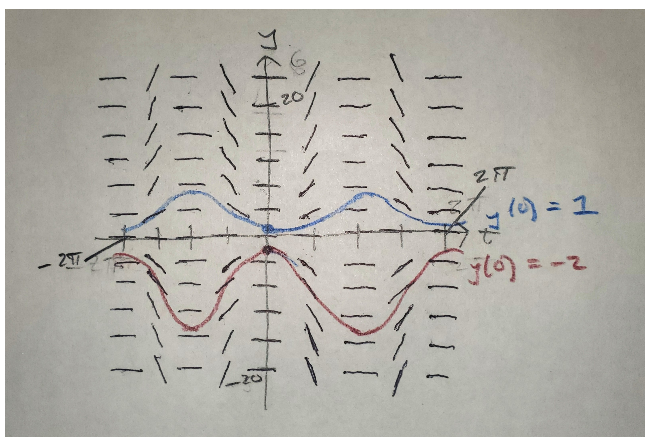

Figure 2.1.3: Slope field and solution curves of $y' = y \sin t$.

Definition 2.1.3 (Autonomous ODE)**.** *An ordinary differential equation that does not contain the independent variable is called* ***autonomous****.*

Examples of autonomous ODEs include the growth/decay model

$$\frac{\mathrm{d}A}{\mathrm{d}t} = kA$$

and the SI model

$$\frac{\mathrm{d}I}{\mathrm{d}t} = kI(n + 1 - I).$$

Self-Check

Write an example of a nonautonomous ODE.

Autonomous ODEs are important also because their slope fields have a special property. Consider the first-order autonomous ODE

$$y' = f(y). \tag{2.1.5}$$

Since the RHS of this ODE is independent of t, the slopes that we sketch on the slope field have nothing to do with t. That is, at the same value of y, the slopes at any two times t_1 and t_2 are the same.

For example, consider again our old friend, $y' = y$. Note that, for any $t \in \mathbb{R}$, the slope of the tangent line to any solution is given by

$$y'(t) = y.$$

That is, the slope is completely constant in t. This is illustrated in the slope field below:

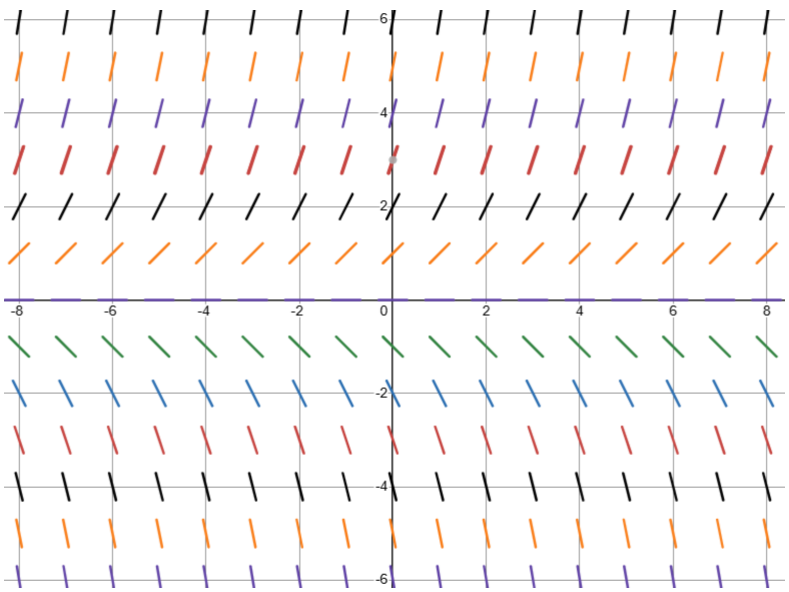

Figure 2.1.4: Slope field of $y' = y$.

Upshot

Slope fields of **autonomous ODEs** are characterized by having the **same slope** at the same y for any t.

Critical points and equilibrium solutions

You may have noticed by now that the zeros of the rate function $f(y)$ are of special importance for autonomous ODEs. This is because by the above, if $f(y(t)) = 0$ for any time t, then $f(y(t)) = 0$ for all t, since

$$y'(t) = f(y(t)) = 0$$

whenever $f(y(t)) = 0$. The zeros of f are called *critical points* of the ODE.

Definition 2.1.4 (Critical point)**.** *A number c is said to be a **critical point** or **stationary point** of the autonomous ODE $y' = f(y)$ if $f(c) = 0$.*

Consider the ODE $y' = f(y)$, and suppose that c is a critical point, so $f(c) = 0$. Thus if we let $y(t) = c$ be the constant function equal to c for all t, then y is a (constant) solution of the ODE. Constant solutions to ODEs are called *equilibrium solutions*.

Definition 2.1.5 (Equilibrium solution)**.** *A constant solution $y(t) = c$ of* ***equilibrium solution****.*

For example, the ODE $y' = y$ has the equilibrium solution $y(t) = 0$. This is indicated by the fact that the slopes in the slope field of $y' = y$ at $y = 0$ are all horizontal (cf. Figure 2.1.4).

Upshot

If $f(c) = 0$, then $y(t) = c$ is an **equilibrium solution** of the ODE $y' = f(y)$.

Self-Check

Suppose that $y(t) = c$ is an equilibrium solution of the ODE $y' = f(y)$, where $f(y)$ and $\frac{\partial f}{\partial y}$ are continuous. Suppose that $u(t)$ is any other nonconstant solution. Can there be some t where $u(t) = c$? Why or why not?

Critical points of ODEs are classified according to their *stability*. If $y(t) = c$ is an equilibrium solution, other nonconstant solutions can either tend *toward* the equilibrium (like in decay), or tend *away* from the equilibrium (like in growth, see Figure 2.1.4). If solutions tend toward the equilibrium, we say that the critical point c is (asymptotically) *stable*, and if solutions tend away, we say that c is *unstable*. There is also a notion of semi-stability (stable on one side and unstable on the other).

Definition 2.1.6 (Stable)**.** *A critical point c of $y' = f(y)$ is said to be* ***asymptotically stable*** *if*

$$\lim_{t \to +\infty} y(t) = c$$

for all solutions starting from an initial point (x_0, y_0) with y_0 "near" c.

Definition 2.1.7 (Unstable)**.** *A critical point c of $y' = f(y)$ is said to be* ***unstable*** *if*

$$\lim_{t \to +\infty} y(t) \neq c$$

for all solutions starting from an initial point (x_0, y_0) with y_0 "near" c.

Self-Check

For $y' = y$, is the equilibrium solution $y = 0$ stable or unstable?

Phase portraits

Since slope fields of autonomous ODEs have the same slope for all t (cf. Figure 2.1.4), we usually prefer to not plot the whole slope field. Instead, we usually just care if solutions are *increasing* or *decreasing*. Thus, since $y' = f(y)$, we really only have to check where $f(y) > 0$ or $f(y) < 0$.

> **Upshot**
>
> For the ODE $y' = f(y)$:
>
> - If $f(y) > 0$, solutions are **increasing**.
> - If $f(y) < 0$, solutions are **decreasing**.

For example, for the ODE $y' = y$, solutions are increasing for y in $(0, +\infty)$ and decreasing for y in $(-\infty, 0)$. We can indicate this with a **phase portrait**, which is a diagram of when solutions to autonomous ODEs are increasing/decreasing. To construct this diagram, think about compressing the slope field to the y-axis and then remember we only care about the *sign* of the slope at each y. The phase portrait for $y' = y$ is shown below.

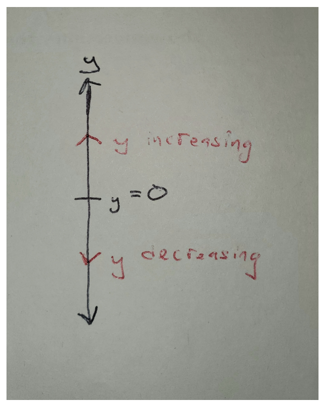

Figure 2.1.5: Phase portrait of $y' = y$.

We also note that the fact that $y = 0$ is an *unstable* equilibrium is evident from the phase portrait, as the arrows point away from the equilibrium. Stable equilibria have arrows pointing toward the solution.

An updated population model

In Section 1.3 we derived the population growth model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP,$$

where a is some constant of proportionality. We also know that this is a simplified model because the population grows exponentially without bound, which assumes unlimited resources, no deaths, etc.

In reality, population is limited by the number of resources. Consider the ODE

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(a - bP). \tag{2.1.6}$$

When P is small, $-bP$ is negligible, so $\frac{\mathrm{d}P}{\mathrm{d}t} \approx aP$ and this looks like exponential growth. However, as P increases, the $-bP$ term becomes significant, and lowers the growth rate.

In the following example we apply many of the concepts of autonomous equations just presented to the ODE (2.1.6). (A side note that ODE (2.1.6) is called the *logistic growth model*).

Example 2.1.4. Consider the logistic growth model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(a - bP).$$

(1) Find all critical points/equilibrium solutions.

(2) Sketch a phase portrait of the ODE.

(3) Suppose that $P(0) = P_0$. What can you say about $\lim_{t\to+\infty} P(t)$ if: $P_0 < 0$? $P_0 \in (0, a/b)$? $P_0 > a/b$?

(4) Classify each of the equilibrium points in (1) as stable, unstable, or semi-stable.

Solution. (1) Let $f(P) = P(a - bP)$. The equilibrium solutions are functions $P = c$ where $f(c) = 0$. We see

$$0 = P(a - bP) \implies P = 0 \text{ or } a - bP = 0 \implies P = 0 \text{ or } P = \frac{a}{b}.$$

Thus the equilibrium solutions are $P = 0$, $P = a/b$.

(2) This is very similar to the process of finding where a function is increasing/decreasing in Calculus I. Note:

- if $P < 0$: $P(a - bP) = (-)(+) \implies \frac{\mathrm{d}P}{\mathrm{d}t} < 0 \implies P$ is decreasing.
- if $0 < P < a/b$: $P(a - bP) = (+)(+) > 0 \implies \frac{\mathrm{d}P}{\mathrm{d}t} > 0 \implies P$ is increasing.
- if $P > a/b$: $P(a - bP) = (+)(-) < 0 \implies \frac{\mathrm{d}P}{\mathrm{d}t} < 0 \implies P$ is decreasing.

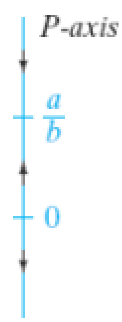

Figure 2.1.6: Phase portrait of $\frac{\mathrm{d}P}{\mathrm{d}t} = P(a - bP)$.

(3) Our solution in (2) says that if $P < 0$ anywhere, then P decreases without bound. Thus

$$P_0 < 0 \implies \lim_{t\to+\infty} P(t) = -\infty.$$

If $0 < P < a/b$, then the phase portrait says that P is increasing but bounded above by a/b (since solutions *cannot* pass through equilibrium solutions). Thus

$$P_0 \in (0, a/b) \implies \lim_{t\to+\infty} P(t) = \frac{a}{b}.$$

Lastly, if $P > a/b$, then the phase portrait says P is decreasing but bounded below by a/b. Hence,

$$P_0 > a/b \implies \lim_{t\to+\infty} P(t) = \frac{a}{b}.$$

(4) The equilibrium solution $P = 0$ is unstable since solutions below 0 diverge to $-\infty$ and solutions above 0 tend to a/b. The equilibrium solution $P = a/b$ is stable since solutions below a/b and above a/b tend to a/b as $t \to +\infty$. □

Summary

- For ODEs in **normal form** $y' = f(t, y)$, the **slope** of any solution $u(t)$ is given by the **rate function** f: $u'(t) = f(t, u(t))$.
- **Slope fields** tell us what solutions to ODEs and IVPs might look like and how they might behave even if we can't solve them explicitly.
- **Autonomous ODEs** are ODEs which do not involve the independent variable: $y' = f(y)$.
- Because autonomous ODEs independent of t, the slope fields of autonomous ODEs are *constant* in t.
- **Equilibrium solutions** of autonomous ODEs are **constant solutions** of the ODE.
- The **logistic growth model** was introduced as a population model accounting for **carrying capacity/resource limitation**.

2.2 Separable equations

Learning objectives

1. Define and identify a **separable ODE**.
2. **Solve** IVPs consisting of separable ODEs.
3. Know how to check for **equilibrium solutions** of separable ODEs; determine if IVPs involving separable equations have **unique solutions**.

Preliminaries and definitions

We are finally about to learn our first solution method in this section. This method applies to all *separable* first-order ODEs. We discuss what this means below.

Consider the first-order ODE $y' = f(t,y)$. A *separable* ODE is one where $f(t,y)$ has a certain form: $f(t,y) = a(t)b(y)$. That is, we can write f as the *product* of a function of t and a function of y.

Definition 2.2.1 (Separable equation)**.** *A first-order ODE of the form*

$$y' = a(t)b(y) \tag{2.2.1}$$

is said to be ***separable****.*

Example 2.2.1. Determine if the following first-order ODEs are separable:

(1) $\frac{\mathrm{d}I}{\mathrm{d}t} = kI(N+1-I)$

(2) $\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-2t}\sin(2t) + e^{-t}\cos^2 t$

(3) $y' = t^2 + y^2$

(4) $y' = (t+2)t^2y^2e^{-3t+2y}$

Solution. (1) Separable, let $a(t) = k$ and $b(I) = I(N+1-I)$.

(2) Separable, let $a(t) = e^{-2t}\sin(2t) + e^{-t}\cos^2 t$, $b(y) = 1$.

(3) Not separable since $t^2 + y^2$ does not factor like $a(t)b(y)$.

(4) Separable. Note that we can write

$$e^{-3t+2y} = e^{-3t}e^{2y},$$

so

$$(t+2)t^2y^2e^{-3t+2y} = (t+2)t^2y^2e^{-3t}e^{2y} = \left[(t+2)t^2e^{-3t}\right]\left[y^2e^{2y}\right],$$

so we can let $a(t) = (t+2)t^2e^{-3t}$, $b(y) = y^2e^{2y}$. □

Upshot

Autonomous ODEs $y' = b(y)$ and **purely time-dependent** ODEs $y' = a(t)$ are **separable**.

Method of solution

Consider the separable ODE from (2.2.1):

$$\frac{dy}{dt} = a(t)b(y). \tag{2.2.2}$$

Note that if $b(y) = 0$, then $y' = 0$ and so $y = C$ for some C. Now suppose that $b(y)$ is not always zero. Then we can divide ODE (2.2.2) by $b(y)$ to get

$$\frac{1}{b(y)}\frac{dy}{dt} = a(t) \quad (b(y(t)) \neq 0).$$

Writing $p(y) = \frac{1}{b(y)}$, this is equal to

$$p(y)\frac{dy}{dt} = a(t). \tag{2.2.3}$$

Now let $y = u(t)$ be a solution of (2.2.3). Thus

$$p(u(t))u'(t) = a(t) \tag{2.2.4}$$

Now integrate both sides of ODE (2.2.4). On the RHS, we get

$$\int a(t)\,dt = A(t) + C_1,$$

where $A(t)$ is an antiderivative of $a(t)$. On the LHS, this is u-substitution:

$$\int p(u(t))u'(t)\,dt = \int p(y)\,dy,$$

since $y = u(t)$ and therefore $dy = u'(t)\,dt$. Hence, if $P(y)$ is an antiderivative of $p(y)$, then by the above we get

$$P(y) + C_2 = \int p(u(t))u'(t)\,dt = \int a(t)\,dt = A(t) + C_1 \implies P(y) = A(t) + (C_1 - C_2).$$

We might as well just write $C = C_1 - C_2$. Hence, we can solve the ODE (2.2.3) by solving

$$P(y) = A(t) + C \tag{2.2.5}$$

for y.

Upshot

Separable ODEs $\frac{dy}{dt} = a(t)b(y)$ can be solved by dividing both sides by $b(y)$:

$$\frac{1}{b(y)}\,dy = a(t)\,dt$$

and then integrating:

$$\int \frac{1}{b(y)}\,dy = \int a(t)\,dt.$$

Examples

Example 2.2.2. *Solve the following IVPs. Be sure to state the interval of existence:*

1. $\frac{dy}{dt} = -y\cos t,\ y(\pi/2) = e^2$
2. $\frac{dy}{dt} = \frac{1}{y}(-e^{-t} + \sin(-2t)),\ y(0) = 2$

Solution. (1) We have

$$\begin{aligned}\frac{1}{y}\,dy = -\cos t\,dt &\implies \int \frac{1}{y}\,dy = \int -\cos t\,dt\\ &\implies \ln|y| = \sin t + C\\ &\implies |y| = e^{\sin t + C}\\ &\implies |y| = C_1 e^{\sin t}\\ &\implies y = C_2 e^{\sin t}.\end{aligned}$$

Applying the initial condition,

$$e^2 = y\left(\frac{\pi}{2}\right) = C_2 e^{\sin\frac{\pi}{2}} = C_2 e^1 = C_2 e \implies C_2 = e.$$

Hence,

$$y(t) = e^{\sin t + 1}, \quad (t \in \mathbb{R}).$$

(2) We find

$$\begin{aligned}y\,dy = -e^{-t} + \sin(-2t)\,dt &\implies \int y\,dy = \int -e^{-t} + \sin(-2t)\,dt\\ &\implies y^2 = e^{-t} + \cos(-2t) + C\\ &\implies y = \sqrt{e^{-t} + \cos(-2t) + C}.\end{aligned}$$

Applying $y(0) = 2$, we obtain

$$2 = y(0) = \sqrt{e^0 + \cos 0 + c} = \sqrt{2 + C} \implies C = 2.$$

Consequently

$$y(t) = \sqrt{e^{-t} + \cos(-2t) + 2}, \quad (t \in \mathbb{R}).$$

□

Example 2.2.3. *Solve the IVP*

$$\frac{dy}{dt} = \sqrt{t}(y^2 - 1), \quad y(0) = -1. \tag{2.2.6}$$

Solution. Dividing, we get

$$\frac{1}{y^2 - 1}\,dy = \sqrt{t}\,dt. \tag{2.2.7}$$

Performing partial fraction decomposition on the LHS,

$$\frac{1}{y^2 - 1} = \frac{A}{y - 1} + \frac{B}{y + 1} \implies 1 = A(y + 1) + B(y - 1),$$

$$\begin{aligned}
y = -1 &\implies B = \frac{-1}{2}, \\
y = 1 &\implies A = \frac{1}{2}, \\
&\implies \frac{1}{y^2-1} = \frac{1/2}{y-1} - \frac{1/2}{y+1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int \frac{1/2}{y-1} - \frac{1/2}{y+1}\,\mathrm{d}y = \int \sqrt{t}\,\mathrm{d}t &\implies \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = \frac{2}{3}t^{\frac{3}{2}} + C \\
&\implies \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{2}{3}t^{\frac{3}{2}} + C \\
&\implies \ln\left|\frac{y-1}{y+1}\right| = \frac{4}{3}t^{\frac{3}{2}} + C \\
&\implies \frac{y-1}{y+1} = e^{\frac{4}{3}t^{3/2}+C} = Ce^{\frac{4}{3}t^{3/2}} \\
&\implies y - 1 = C(y+1)e^{\frac{4}{3}t^{3/2}} \\
&\implies y - Cye^{\frac{4}{3}t^{3/2}} = 1 + Ce^{\frac{4}{3}t^{3/2}} \\
&\implies y = \frac{1+Ce^{\frac{4}{3}t^{3/2}}}{1-Ce^{\frac{4}{3}t^{3/2}}}.
\end{aligned}$$

Then, applying the initial condition $y(0) = 1$, we find

$$-1 = y(0) = \frac{1+C}{1-C} \implies -(1-C) = 1 + C \implies C - 1 = C + 1 \implies 2 = 0. \quad \rightarrow\!\leftarrow$$

The issue is that $y = \pm 1$ are the two equilibrium solutions of the ODE (2.2.6), and we divided by these in (2.2.6). In this case, $y = -1$ is the unique (singular) solution of the IVP (2.2.6) (check this with Theorem 1.2.1). That is, we cannot obtain this solution for any value of C above (in fact we get a contradiction if we try). □

Upshot

Check for **equilibrium solutions** of IVPs *before* applying a solution method.

Example 2.2.4. *Solve the following IVPs. Make sure to state the intervals of existence:*

(1) $y' = \frac{t}{y}, \quad y(0) = -1$

(2) $y' = (y-1)^2, \quad y(0) = 0$

Solution. (1) First, there are no equilibrium solutions becuase there is no value of y such that $f(t,y) = t/y$ satisfies $f(t,y) = 0$ for all t. Second, notice that this IVP has a unique solution by Theorem 1.2.1, since the only "bad" point is $y = 0$ and the initial condition is $y = -1$.

We first separate variables:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y} \implies y\,\mathrm{d}y = t\,\mathrm{d}t$$

$$\begin{aligned}&\implies \int y\,\mathrm{d}y = \int t\,\mathrm{d}t\\&\implies \frac{1}{2}y^2 = \frac{1}{2}t^2 + C\\&\implies y^2 = t^2 + C\\&\implies y = \pm\sqrt{t^2 + C}.\end{aligned}$$

(The $\pm$ is because $y^2 = t^2 + C$ can have two solutions, depending on the sign of y. In this case it is negative since the initial condition is negative.)

Applying the initial condition $y(0) = -1$, we see:

$$-1 = y(0) = -\sqrt{C} \implies C = 1.$$

Hence,

$$y(t) = -\sqrt{t^2+1} \quad (t \in \mathbb{R}).$$

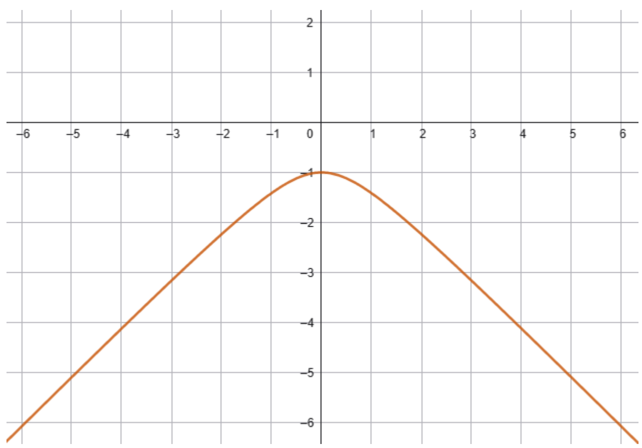

Figure 2.2.1: $y(t) = -\sqrt{t^2+1}$.

(2) First, $y = 1$ is an equilibrium solution of the ODE, but it does not solve the IVP since the initial condition is at 0. Notice that Theorem 1.2.1 guarantees a unique solution otherwise.

Separating variables,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} = (y-1)^2 &\implies \frac{\mathrm{d}y}{(y-1)^2} = \mathrm{d}t\\&\implies \int \frac{\mathrm{d}y}{(y-1)^2} = \int \mathrm{d}t\\&\implies -\frac{1}{y-1} = t + C\\&\implies y - 1 = -\frac{1}{t+C}\\&\implies y = 1 - \frac{1}{t+C}.\end{aligned}$$

Applying $y(0) = 0$, we get

$$0 = y(0) = 1 - \frac{1}{C} \implies C = 1.$$

Consequently

$$y(t) = 1 - \frac{1}{t+1} \quad (t \in (-1, +\infty)).$$

□

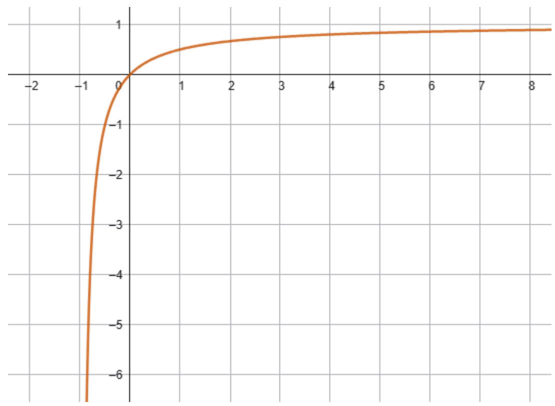

Figure 2.2.2: $y(t) = 1 - \frac{1}{t+1}$.

Summary

- (First-order) **Separable ODEs** are those of the form
$$y' = a(t)b(y).$$
- The **solution method** of separable ODEs was discussed. Roughly, we "cross-multiply" so that all y terms are on the left and all t terms are on the right, and then integrate.
- Recall that **equilibrium solutions** of separable ODEs are values y such that
$$y' = a(t)b(y) = 0,$$
that is, values y such that $b(y) = 0$. Know to check for these before applying a solution method.
- Techniques from Chapter 1 were applied to determine **existence and uniqueness of solutions** and **intervals of existence** of solutions to IVPs consisting of separable ODEs.

2.3 Linear equations

Learning objectives

1. Define and identify a first-order **linear ODE**.
2. Learn how to put a first-order linear ODE into **standard form**.
3. Solve linear ODEs using the method of **integrating factors**; explain how integrating factors help with solving linear ODEs.

Preliminaries and definitions

The next class of equations we will deal with are the *linear* equations. Linear ODEs are important first of all because many fundamental physical laws are linear (e.g., $F = ma$, $\sigma = E\varepsilon$), but also because the linear equations are frequently considered the “broadest" class of equations which can be solved exactly. In fact, nonlinear ODEs are often approximated with linear ODEs using a process called “linearization" as a first step toward understanding solution properties and behavior.

In this section we consider the solution of IVPs consisting of first-order linear ODEs. Remember from Section 1.1 that the “linearity" here means that there are no nonlinear functions of y which appear in the ODE (nonlinear functions of t are OK).

Definition 2.3.1 (First-order linear ODE)**.** *A* ***first-order linear ODE*** *is an equation of the form*

$$a(t)y' + b(t)y = g(t). \tag{2.3.1}$$

Note that, in (2.3.1), if $a(t) = 0$ for all t, then $y(t) = g(t)/b(t)$ and there is no ODE to solve. So, we can divide by $a(t)$ to get a more useful form of linear ODEs when it comes to finding solutions.

Definition 2.3.2 (Standard form)**.** *A first-order linear ODE in* ***standard form*** *is given by*

$$y' + p(t)y = f(t). \tag{2.3.2}$$

Upshot

First-order linear ODEs are those which can be put into the form

$$y' + p(t)y = f(t).$$

Example 2.3.1. *Determine if the following equations are linear. Are they also separable?*

(1) $y' = -4y\sin t$

(2) $y' = -4t\sin y$

(3) $y' = 1 - 4y$

(4) $y' = t + y$

Solution. (1) Linear ($p(t) = 4\sin t$, $f(t) = 0$) and separable ($y' = (-4y)(\sin t)$).

(2) Nonlinear now because we have the $\sin y$, which is nonlinear in y. Still separable.

(3) Linear ($p(t) = 4$, $f(t) = 1$) and separable ($y' = (1 - 4y)(1)$).

(4) Linear ($p(t) = -1$, $f(t) = t$) but not separable. □

Method of solution

Consider again the standard form of a first-order linear ODE:

$$y' + p(t)y = f(t). \tag{2.3.3}$$

The idea is to multiply both sides of (2.3.3) by some choice of function I and apply product rule on the LHS.

We first must determine what the function "I" should be. Multiply both sides of (2.3.3) by $I(t)$ to get

$$I(t)y' + I(t)p(t)y = I(t)f(t). \tag{2.3.4}$$

The idea here is if we can make the LHS of (2.3.4) look like $\frac{\mathrm{d}}{\mathrm{d}t}[I(t) \cdot y]$, then we can just integrate both sides of (2.3.4). Thus, we want to choose $I(t)$ such that

$$I(t)y' + I(t)p(t)y = \frac{\mathrm{d}}{\mathrm{d}t}[I(t) \cdot y] = I(t)y' + I'(t)y. \tag{2.3.5}$$

Subtracting both sides of (2.3.5) by $I(t)y'$ and then dividing by y (assuming that y is not always 0) gives

$$\frac{\mathrm{d}I}{\mathrm{d}t} = I(t)p(t). \tag{2.3.6}$$

Notice that the ODE (2.3.6) is a separable equation in I. Separating variables, we find

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} = p(t)I &\implies \frac{\mathrm{d}I}{I} = p(t)\,\mathrm{d}t \\ &\implies \int \frac{\mathrm{d}I}{I} = \int p(t)\,\mathrm{d}t \\ &\implies \ln|I| = \int p(t)\,\mathrm{d}t \\ &\implies I(t) = Ce^{\int p(t)\,\mathrm{d}t}. \end{aligned} \tag{2.3.7}$$

Now we may choose any value of C in (2.3.7) (again, the goal for now was only to make equation (2.3.5) hold, and this holds regardless of the value of C). For simplicity, we choose $C = 1$, and define

$$I(t) = e^{\int p(t)\,\mathrm{d}t}. \tag{2.3.8}$$

Upshot

With $I(t)$ as in (2.3.8), ODE (2.3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}[I \cdot y] = I(t)f(t).$$

The point of this is the following. We have found a function I such that (2.3.5) holds. This means

$$I(t)y' + I(t)p(t)y = I(t)y' + I'(t)y = \frac{\mathrm{d}}{\mathrm{d}t}[I \cdot y].$$

Inserting this into (2.3.4), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[I \cdot y] = I(t)f(t). \tag{2.3.9}$$

Integrating both sides then gives

$$I \cdot y = \int I(t) f(t) \, \mathrm{d}t. \tag{2.3.10}$$

Finally, dividing (2.3.10) by $I(t)$ (assuming $I(t)$ is not always 0) gives the solution y to ODE (2.3.3):

$$y(t) = \frac{1}{I(t)} \int I(t) f(t) \, \mathrm{d}t \quad \left(I(t) = e^{\int p(t)\, \mathrm{d}t}\right). \tag{2.3.11}$$

Before working through some examples, we first define some terminology related to the solution method.

Definition 2.3.3 (Integrating factor)**.** *The function $I(t) = e^{\int p(t)\, \mathrm{d}t}$ as defined in (2.3.8) is called an **integrating factor** for the ODE (2.3.3).*

We call I an *integrating factor* because it helps us integrate both sides of ODE (2.3.3) (cf. steps (2.3.9), (2.3.10)).

Upshot

To solve a first-order linear ODE:

1. Put the equation into standard form (2.3.3): $y' + p(t)y = f(t)$.
2. Identify the function $p(t)$ and find the integrating factor $I(t) = e^{\int p(t)\, \mathrm{d}t}$ (no constant of integration necessary here).
3. Multiply both sides of the ODE by $I(t)$. The LHS is automatically equal to $\frac{\mathrm{d}}{\mathrm{d}t}[I \cdot y]$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}[I \cdot y] = I(t) f(t).$$

4. Integrate both sides to get

$$I \cdot y = \int I(t) f(t) \, \mathrm{d}t,$$

and then divide both sides by $I(t)$ to isolate y (remember the $+C$ here).

Examples

Example 2.3.2. *Solve the following IVPs:*

(1) $y' + ty = (1-t)e^{-t}, \quad y(0) = 1$

(2) $y' + y = -\cos t, \quad y(\pi) = \frac{1}{2}$

Solution. (1) First, this ODE is already in standard form. Here $p(t) = t$ and $f(t) = (1-t)e^{-t}$. We find the integrating factor:

$$I(t) = e^{\int p(t)\, \mathrm{d}t} = e^{\int t\, \mathrm{d}t} = e^{\frac{1}{2}t^2}.$$

Next, multiply both sides of the ODE by $e^{t^2/2}$:

$$e^{\frac{t^2}{2}} y' + t e^{\frac{t^2}{2}} y = (1-t) e^{\frac{t^2}{2} - t}.$$

We can write the LHS as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\frac{t^2}{2}}y\right] = (1-t)e^{\frac{t^2}{2}-t}.$$

Integrating both sides,

$$\begin{aligned} e^{\frac{t^2}{2}}y &= \int \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\frac{t^2}{2}}y\right]\mathrm{d}t = \int (1-t)e^{\frac{t^2}{2}-t}\,\mathrm{d}t \\ &= \int -e^u\,\mathrm{d}u \end{aligned}$$

(by u-substitution, $u = \frac{t^2}{2} - t$)

$$= -e^u + C = -e^{\frac{t^2}{2}-t} + C.$$

Hence, dividing by the integrating factor $e^{\frac{t^2}{2}}$,

$$y = \left(-e^{\frac{t^2}{2}-t} + C\right)e^{-\frac{t^2}{2}} = -e^{-t} + Ce^{-\frac{t^2}{2}}.$$

Finally, applying the initial condition $y(0) = 1$, we get

$$1 = y(0) = -e^0 + Ce^0 = -1 + C \implies C = 2.$$

Thus

$$y(t) = -e^{-t} + 2e^{-\frac{t^2}{2}}.$$

(2) This ODE is again in standard form. Here $p(t) = 1$ and $f(t) = -\cos t$, so

$$I(t) = e^{\int p(t)\,\mathrm{d}t} = e^{\int \mathrm{d}t} = e^t.$$

Multiplying both sides by $I(t) = e^t$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[e^t y] = -e^t\cos t.$$

Integrating,

$$e^t y = \int \frac{\mathrm{d}}{\mathrm{d}t}[e^t y]\,\mathrm{d}t = \int -e^t\cos t\,\mathrm{d}t.$$

We have to integrate $\int -e^t\cos t\,\mathrm{d}t$ by parts. Put $S = \int -e^t\cos t\,\mathrm{d}t$. We calculate

$$\begin{aligned} S &= \int -e^t\cos t\,\mathrm{d}t \\ &= -e^t\cos t - \int e^t\sin t\,\mathrm{d}t \end{aligned}$$

(integrating by parts, letting $u = -\cos t$, $\mathrm{d}v = e^t\,\mathrm{d}t$)

$$= -e^t\cos t - \left(e^t\sin t - \int e^t\cos t\,\mathrm{d}t\right)$$

(integrating by parts again, letting $u = \sin t, \mathrm{d}v = e^t\,\mathrm{d}t$)

$$= -e^t(\cos t + \sin t) - \left(\int -e^t \cos t\,\mathrm{d}t\right)$$
$$= -e^t(\cos t + \sin t) - S.$$

Adding S to both sides and then dividing by 2, we find

$$S = \int e^{-t} \cos t\,\mathrm{d}t = -\frac{1}{2}e^t(\cos t + \sin t) + C.$$

Hence, we have

$$e^t y = \int -e^t \cos t\,\mathrm{d}t = -\frac{1}{2}e^t(\cos t + \sin t) + C \implies y = \left(-\frac{1}{2}e^t(\cos t + \sin t) + C\right)e^{-t}$$
$$\implies y = -\frac{1}{2}(\cos t + \sin t) + Ce^{-t}.$$

Lastly, applying the initial condition $y(\pi) = \frac{1}{2}$, we obtain

$$\frac{1}{2} = y(\pi) = -\frac{1}{2}(\cos \pi + \sin \pi) + Ce^{-\pi} = \frac{1}{2} + Ce^{-\pi} \implies C = 0.$$

Consequently

$$y(t) = -\frac{1}{2}(\cos t + \sin t).$$

□

Self-Check

Would you rather solve a separable ODE or a linear ODE? Why?

Example 2.3.3. *Solve the IVP* $ty' + 9y = 0$, $y(1) = 1$.

Solution. We first divide by t to put the ODE in standard form:

$$y' + 9\frac{y}{t} = 0.$$

Here $p(t) = \frac{9}{t}$ and $f(t) = 0$. So

$$I(t) = e^{\int p(t)\,\mathrm{d}t} = e^{\int \frac{9}{t}} = e^{9 \ln t} = t^9.$$

We multiply both sides by $I(t) = t^9$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}[t^9 y] = 0t^9 = 0.$$

Integrating both sides,

$$t^9 y = \int \frac{\mathrm{d}}{\mathrm{d}t}[t^9 y]\,\mathrm{d}t = C.$$

Dividing by t^9,

$$y(t) = \frac{C}{t^9}.$$

Finally, applying the initial condition $y(1) = 1$, we find

$$1 = y(1) = \frac{C}{1} \implies C = 1.$$

Hence,

$$y(t) = \frac{1}{t^9} \quad (t \in (0, +\infty))$$

□

Notice that we could have also solved the ODE in Example 2.3.3 by separating variables:

$$ty' + 9y = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{9y}{t} \implies \frac{\mathrm{d}y}{y} = -\frac{9}{t}\,\mathrm{d}t.$$

The amount of work involved here is really about the same (algebraic manipulation for the setup, 2 integrations, and then apply the initial condition). Most students find solving separable equations simpler, though, so it is usually best to check if the ODE is separable as an early step.

Summary

- First-order **linear ODEs** are those with no nonlinear functions of y' or y. First-order linear ODEs in **standard form** are given by

$$y' + p(t)y = f(t).$$

- The concept of **integrating factors** was introduced as a tool to help us integrate both sides of a linear ODE; an integrating factor for the ODE $y' + p(t)y = f(t)$ is given by

$$I(t) = e^{\int p(t)\,\mathrm{d}t}.$$

- The integrating factor method for solving linear ODEs was discussed. Roughly, we first find an integrating factor $I(t)$ for the ODE, multiply the ODE by $I(t)$, integrate both sides, and then solve for y.

The end-of-section summaries will now be followed by a "method checklist." This is a list of methods that you have learned for solving ODEs, in ascending order of difficulty (at least in my opinion). Usually, I recommend attempting to use a method toward the top and then working down.

Method Checklist

(1) Guessing/equilibrium solutions/direct integration

(2) Separable ODE? Separation of variables (Section 2.2)

(3) Linear ODE? Integrating factors (Section 2.3)